Micromechanics and structural response of functionally graded, particulate-matrix, fiber-reinforced composites

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\textbf{A B S T R A C T}

Reinforcement of fibrous composites by stiff particles embedded in the matrix offers the potential for simple, economical functional grading, enhanced response to mechanical loads, and improved functioning at high temperatures. Here, we consider laminated plates made of such a material, with spherical reinforcement tailored by layer. The moduli for this material lie within relatively narrow bounds. Two separate moduli estimates are considered: a "two-step" approach in which fibers are embedded in a homogenized particulate matrix, and the Kanaun–Jeulin (Kanaun, S.K., Jeulin, D., 2001. Elastic properties of hybrid composites by the effective field approach. Journal of the Mechanics and Physics of Solids 49, 2339–2367) approach, which we re-derive in a simple way using the Benveniste (1988) method. Optimal tailoring of a plate is explored, and functional grading is shown to improve the performance of the structures considered. In the example of a square, simply supported, cross-ply laminated panel subjected to uniform transverse pressure, a modest functional grading offers significant improvement in performance. A second example suggests superior blast resistance of the panel achieved at the expense of only a small increase in weight.

1. Introduction

Much of Liviu Librescu’s research career was devoted to the science of developing improved structural systems through novel structural and material reinforcement schemes (Librescu, 1976; Librescu and Song, 2006). In this spirit, we contribute an article to this special edition of the International Journal of Solids and Structures devoted to his memory that focuses on optimizing a lightweight, dynamically loaded composite panel through careful selection of material properties and a novel reinforcing scheme.

Hybrid composite materials consisting of an isotropic polymer matrix reinforced by both particles and unidirectional fibers offer the potential for simple functional grading to tailor mechanical response and reduce stress concentrations around attachments and discontinuities. Local particle reinforcement can increase stiffness and strength at key locations at the expense of a relatively small increase in weight. Moreover, polymeric composite structures subjected to thermal loading exhibit matrix deterioration at high temperatures, but a matrix reinforced with ceramic particles offers the potential to increase structural endurance and load-carrying capacity at high temperatures (Birman and Byrd, 2007).

Effective application of hybrid particulate-matrix fiber-reinforced composites involves analysis of their stiffness in structural systems with either uniform or variable property distributions. For example, these structures may represent...
thin-walled shells or plates with in-surface variable stiffness. A particularly simple and attractive grading scheme involves embedding particles only in the outer layers of a laminate, achieving maximal increases in bending stiffness with a minimum of additional weight. Tailoring the volume fraction of particles, especially on the outer laminae of a composite, is far simpler than the alternative of varying fiber volume fraction or fiber orientations within a lamina.

A broad range of homogenization methods exists to predict the properties of composite materials containing fibrous or particulate inclusions (e.g., Tucker and Liang, 1999; Torquato, 2001; Kakavas and Kontoni, 2006). The accuracy of these methods has been the subject of numerous investigations. Hu and Weng (2000) compare micromechanical models, including the double-inclusion method (Hori and Nemat-Nasser, 1993), the models of PonteCastaneda and Willis (1995), the model of Kuster and Toksöz (1974), and the Mori–Tanaka model (1973), with each model showing ranges of accuracy. The Mori–Tanaka approach, applied in Benveniste’s (1987) generalized form in this paper, is considered accurate for predicting the elastic moduli of particulate-reinforced matrices only for volume fractions of inclusions below 40% (Kwon and Dharan, 1995; Sun et al., 2007). On the other hand, Noor and Shah (1993) showed that the Mori–Tanaka method provides an accurate prediction of the properties of fiber-reinforced composites even at a high volume fraction of fibers. In the following analysis, we study rigorous bounds in conjunction with estimates to assess the degree to which estimates can be trusted.

General approaches to the characterization of a hybrid composite consisting of three different phases have been proposed by Yin et al. (2004) and Kanaun and Jeulin (2001). The Kanaun–Jeulin approach is based upon an effective field method with the assumption that the strain field within individual inclusions can differ for each population. We apply this approach in this paper, and re-derive it through a generalization of the modification of the Mori–Tanaka method suggested by Benveniste (1987); for the stiffness tensor of a three-phase composite, the models yield identical predictions. Many other authors have presented additional techniques for composite materials with multiple classes of reinforcement. Luo and Weng (1989) considered a three-phase concentric cylinder model consisting of a fiber, a coating (“intermediate matrix”), and a matrix, and applied the resulting Eshelby (1957, 1959) tensors through a Mori–Tanaka formulation. The resulting estimates coincided with the lower Hill–Hashin bounds, or fell within them. Benveniste et al. (1989) and Dasgupta and Bhandarkar (1992) considered modifications of Benveniste’s method to model composites with coated inclusions.

Bounds on material properties (e.g., moduli of elasticity, shear moduli, bulk modulus, and Poisson’s ratios) are calculated in this article using techniques summarized in Torquato (2001). Bounds such as those of Weng (1992) and the three-point bounding technique (e.g. Milton and Phan-Thien, 1982), which include descriptions of reinforcement geometry, are far tighter than those of Voigt and Reuss (Hill, 1952), Hashin (1962), or Hashin and Shtrikman (1962, 1963). This paper begins with a re-derivation of the Kanaun–Jeulin estimate for the stiffness tensor of a three-phase composite, based on an extrapolation of the Benveniste approach (1987) for the case of multiple types of inclusions (fibers and particles). Subsequently, bounds and additional estimates are developed in a two-step fashion: bounds and estimates are first employed for the response of a particulate matrix, and these are subsequently employed to establish bounds and estimates for the response of a particulate-reinforced matrix additionally reinforced by unidirectional fibers. Finally, the advantages of embedding a small amount of particles in fiber-reinforced materials are illustrated through consideration of a simply supported cross-ply composite plate subjected to static, instantaneous, or blast pressure loadings. In all of these examples, adding glass particles to selected glass/epoxy laminae within a laminated plate resulted in significant reductions of maximum deformations achieved with very modest additional weight.

2. Re-derivation of the Kanaun–Jeulin estimate using a Benveniste-type approach

Consider a material where two different types of isotropic inclusions are distributed within an isotropic matrix. The properties of the matrix are identified in the subsequent solution with the subscript $i = 1$, while two types of the inclusions are denoted by $i = 2$ and $i = 3$. We take phase 2 to be spherical particles and phase 3 to be aligned fibers. We derive the stiffness tensor of such a material through generalization of Benveniste’s (1987) solution for a particulate composite with a single type of inclusions.

The approach is based on the following assumptions:

1. All material phases are isotropic and linearly elastic.
2. The perturbed strain in the matrix due to the presence of inclusions is not affected by the interaction of the two types of inclusions. In other words, each type of inclusion $i = 2$ and $i = 3$ affect the strains in the matrix, but the perturbed matrix strain due to the interaction between these inclusions is assumed to be of second order.
3. Phase 3 is represented by aligned fibers of circular cross section; i.e., the composite material is a lamina with embedded particles. This assumption is needed to utilize the Eshelby tensor for cylindrical inclusions. In general, the derivation shown below is independent of the orientation and shape of fibers and particles as long as the corresponding Eshelby tensor is known.

The average stress ($\bar{\sigma}_0$) and average strain ($\bar{\epsilon}_0$) tensors for the material under consideration are related through the effective stiffness tensor $E$:

$$\bar{\sigma}_0 = E \bar{\epsilon}_0. \quad (1)$$
where boldface symbols represent tensors and bars denote a volumetric average over the entire composite.

The effective stiffness tensor for a matrix with two different types of embedded inclusions can be written as a generalization of the expression proposed by Hill (1963)

$$\mathbf{C} = \mathbf{L}_1 + f_2(\mathbf{L}_2 - \mathbf{L}_1)\mathbf{A}_2 + f_3(\mathbf{L}_3 - \mathbf{L}_1)\mathbf{A}_3,$$

where \( \mathbf{L}_i \) is the stiffness tensor of the \( i \)-th phase, \( f_2 \) and \( f_3 \) are volume fractions of the corresponding types of inclusions, and the tensors of concentration factors \( \mathbf{A}_i \) and \( \mathbf{A}_3 \) represent the relationships between the tensors of average strains in the corresponding inclusions \( (\bar{\varepsilon}_i) \) and the mean remote strain tensor \( (\bar{\varepsilon}_0) \):

$$\bar{\varepsilon}_2 = \mathbf{A}_2\bar{\varepsilon}_0, \quad \bar{\varepsilon}_3 = \mathbf{A}_3\bar{\varepsilon}_0.$$  \( \tag{3} \)

Note that according to the second assumption, this approach does not explicitly account for the interaction of different types of inclusions. Accordingly, it is applicable only if at least one type of inclusion has a relatively small volume fraction.

Expanding the Mori and Tanaka (1973) ideas, the tensors of average strain in the matrix and in the phases are represented as

$$\bar{\varepsilon}_1 = \bar{\varepsilon}_0 + \bar{\varepsilon}_2 + \bar{\varepsilon}_3,$$

$$\bar{\varepsilon}_2 = \bar{\varepsilon}_0 + \bar{\varepsilon}_2 + \bar{\varepsilon}_3 + \bar{\varepsilon}_2',$$

$$\bar{\varepsilon}_3 = \bar{\varepsilon}_0 + \bar{\varepsilon}_2 + \bar{\varepsilon}_3 + \bar{\varepsilon}_3',$$

where \( \bar{\varepsilon}_i \) are tensors of perturbations superimposed on the average strain in the matrix as a result of the presence of the corresponding inclusions, and \( \bar{\varepsilon}_i' \) are tensors of average perturbed strain in the inclusions relative to the tensor of average strain in the matrix.

The tensors of average stresses in the inclusions can now be expressed in terms of the stiffness of the matrix:

$$\mathbf{L}_2(\bar{\varepsilon}_0 + \bar{\varepsilon}_2 + \bar{\varepsilon}_3 + \bar{\varepsilon}_2') = \mathbf{L}_2(\bar{\varepsilon}_0 + \bar{\varepsilon}_2 + \bar{\varepsilon}_3 + \bar{\varepsilon}_2'),$$

$$\mathbf{L}_3(\bar{\varepsilon}_0 + \bar{\varepsilon}_2 + \bar{\varepsilon}_3 + \bar{\varepsilon}_3') = \mathbf{L}_3(\bar{\varepsilon}_0 + \bar{\varepsilon}_2 + \bar{\varepsilon}_3 + \bar{\varepsilon}_3'),$$  \( \tag{5} \)

where \( \bar{\varepsilon}_i' \) are tensors of average correlation strain in the corresponding type of inclusions. These tensors are related to the tensors of perturbations in the inclusions by

$$\bar{\varepsilon}_i' = \mathbf{S}_i^{-1}\bar{\varepsilon}_i', \quad \bar{\varepsilon}_3' = \mathbf{S}_3^{-1}\bar{\varepsilon}_3',$$  \( \tag{6} \)

where \( \mathbf{S}_i \) are the fourth-order Eshelby tensors, presented in Appendix A for the cases of spherical inclusions and for infinitely long cylindrical inclusions (fibers).

From (4) and (5), the tensors of perturbation strains can be expressed in terms of the tensors of average strain in the corresponding inclusions as

$$\bar{\varepsilon}_i' = \mathbf{S}_i^{-1}(\mathbf{L}_i - \mathbf{L}_1)\bar{\varepsilon}_i \quad (i = 2, 3).$$  \( \tag{7} \)

As also directly follows from (4):

$$\bar{\varepsilon}_1' = \bar{\varepsilon}_1 - \bar{\varepsilon}_1.$$  \( \tag{8} \)

The subsequent transformation requires expression of the tensors of average strain in each type of inclusion in terms of the tensor of the average strain in the matrix; i.e., the linear transformations \( \mathbf{T}_i \) such that

$$\bar{\varepsilon}_i = \mathbf{T}_i\bar{\varepsilon}_1.$$  \( \tag{9} \)

This is easily accomplished using (7) and (8)

$$\mathbf{T}_i = [\mathbf{I} + \mathbf{S}_i\mathbf{L}_1^{-1}(\mathbf{L}_i - \mathbf{L}_1)]^{-1},$$  \( \tag{10} \)

where \( \mathbf{I} \) is a fourth-order unit tensor.

The tensors of concentration factors are now determined by expressing the tensor of the applied strain, which also represents the average strain in an equivalent homogeneous material, in terms of strains in the constituent phases through the rule of mixtures:

$$\bar{\varepsilon}_0 = f_1\bar{\varepsilon}_1 + f_2\bar{\varepsilon}_2 + f_3\bar{\varepsilon}_3.$$  \( \tag{11} \)

Using (3) and (9) in (11) yields

$$\mathbf{A}_i = \mathbf{T}_i(f_1\mathbf{I} + f_2\mathbf{T}_2 + f_3\mathbf{T}_3)^{-1}.$$  \( \tag{12} \)

Using the concentration tensors (12) in the tensor of effective stiffness (2) yields an estimate coinciding with that derived by Kanaun and Jeulin (2001) using the effective field method

$$\mathbf{C} = \mathbf{L}_1 + \sum_{i=2}^3 f_i(\mathbf{L}_i - \mathbf{L}_1)\mathbf{T}_i(f_1\mathbf{I} + f_2\mathbf{T}_2 + f_3\mathbf{T}_3)^{-1}. $$  \( \tag{13} \)
Computations evaluating (13) can be performed simply using the tensor decomposition presented by Walpole (1984) and summarized recently by Sevostianov and Kachanov (2007). Note that for the case of a single type of inclusions this result converges to the formula derived by Benveniste (1987).

3. "Two-step" approach to the bounding and estimation of the elastic response of a particulate-matrix, fiber-reinforced composite material

Bounds and estimates are combined here to evaluate the effective stiffness tensor of an isotropic matrix material reinforced by a random dispersion of identical spherical particles. Subsequently, these bounds and estimates are used to represent the matrix of a fiber-reinforced composite material, and additional bounds and estimates are applied to model this fiber-reinforced composite. An underlying assumption is that the spherical particles are small compared to the fiber radii.

In the example that will be studied, \( \phi_1 \) and \( \phi_2 \) refer to the volume fractions of epoxy and (relatively stiff) identical spherical particles, respectively, within the particulate-reinforced matrix; these relate to the overall volume fractions of epoxy \( (f_1) \) and spherical particles \( (f_2) \) as \( \phi_2 = f_2/(f_1 + f_2) \), with \( \phi_1 = 1 - \phi_2 \). The dense packing limit for identical spherical particles is that \( \phi_2 \) cannot exceed approximately 0.63 (Torquato, 2001).

3.1. Bounds and estimates of the mechanical response of the particle-reinforced matrix

The first step of the "two-step" procedure is to estimate and bound the effective bulk modulus, \( K_{pm} \), and the effective shear modulus, \( G_{pm} \), of the combination of isotropic spherical particles and isotropic matrix. The matrix has bulk modulus \( K_1 \) and shear modulus \( G_1 \), and the spherical particles have bulk modulus \( K_2 \) and shear modulus \( G_2 \).

Beran and Molyneux (1966) and McCoy (1970) obtained three-point bounds on \( K_{pm} \) and \( G_{pm} \), for two-phase composites. These involve two parameters, \( \zeta_2 \) and \( \eta_2 \) (additionally, \( \eta_1 = 1 - \eta_2, \zeta_1 = 1 - \zeta_2 \)) that characterize the shape and distribution of the two phases (e.g. Torquato, 1991). A three-step procedure is computationally expensive, but only a limited number of microstructures have been characterized. Values range between \( 0.15 \phi_2 < \zeta_2 < \phi_2 \) and \( 0.5 \phi_2 < \eta_2 < \phi_2 \) (Torquato, 1991). For randomly spaced spherical particles, \( \zeta_2 \sim 0.211 \phi_2 \) and \( \eta_2 \sim 0.483 \phi_2 \) (Torquato, 2001). The Milton and Phan-Thien improvement on the McCoy (1970) three-point bounds \( G_{pm} \) is

\[
\langle G \rangle - \frac{\phi_1 \phi_2 (G_2 - G_1)^2}{\langle G \rangle + \Theta} \leq G_{pm} \leq \langle G \rangle - \frac{\phi_1 \phi_2 (G_2 - G_1)^2}{\langle G \rangle + \phi \Theta},
\]

where

\[
\Theta = \frac{\langle G \rangle (G_1 + G_2)}{\langle G \rangle} = \frac{6 \langle G \rangle (G_1 + G_2)}{(2K_1 - G_2) + 30 \langle G \rangle},
\]

in which \( \langle \cdot \rangle \) denotes a weighted average (for example, \( \langle G \rangle = G_1 \phi_1 + G_2 \phi_2 \), \( \zeta \leq \zeta_1 \leq \zeta_2 \leq \zeta \), \( \eta = \eta_1 \phi_1 + \eta_2 \phi_2 \)), and a tilde represents a reverse-weighted average, e.g. \( \langle G \rangle \sim \phi_2 G_2 + \phi_1 G_1 \).

Milton’s (1981) form of the three-point Beran and Molyneux (1966) bounds on the bulk modulus of an isotropic two-phase composite is

\[
\langle K \rangle - \frac{\phi_1 \phi_2 (K_2 - K_1)^2}{\langle K \rangle + 2d(d - 1)/d \langle G \rangle} \leq K_{pm} \leq \frac{\phi_1 \phi_2 (K_2 - K_1)^2}{\langle K \rangle + 2d(d - 1)/d \langle G \rangle},
\]

where \( d = 3 \).

The “three-point” estimates lying between these bounds incorporate detailed information on the geometry and distribution of the spherical particles (Torquato, 2001):

\[
\phi_2 \frac{\mu_{k_2}}{K_{k_2}} = 1 - \frac{(d + 2)(d - 1)G_1 K_{k_2} \mu_{k_1} \phi_1 \zeta_2}{d(K_1 + 2G_1)}
\]

and

\[
\phi_2 \frac{\mu_{k_2}}{K_{k_2}} = 1 - \frac{2G_1 K_{k_2} \mu_{k_1} \phi_1 \zeta_2}{d(K_1 + 2G_1)} - \frac{\zeta_2(d^2 - 4)G_1 (2K_1 + 3G_1) + \eta_2(2d - 4G_1)}{2d(K_1 + 2G_1)^2} \mu_{k_1}^2 \phi_1,
\]

where

\[
d = 3, \quad K_{k_2} = K_2 - K_1, \quad K_{k_1} = K_e - K_1, \quad \phi_{k_2} = G_e - G_1, \quad \mu_{k_2} = \frac{G_2 - G_1}{G_2 - B G_1}, \quad \mu_{k_1} = \frac{G_e - G_1}{G_e - B G_1}, \quad A = \frac{2(d - 1)}{d}, \quad \text{and}
\]

\[
B = \frac{d K_1/2 + (d + 1)(d - 2)G_1/d}{K_1 + 2G_1}.
\]
3.2. Bounds on and estimates of the mechanical response of a fiber- and particle-reinforced lamina

We now study the mechanics of a lamina containing aligned fibers of volume fraction $f_f$ embedded in an isotropic matrix of volume fraction $f_m + f_f$. The matrix mechanical properties are those of the homogenized particle-reinforced matrix discussed in Section 3.1; that is, the matrix is an isotropic continuum with bulk modulus $K_{pm}$ and shear modulus $G_{pm}$. The fibers are isotropic with bulk modulus $K_{fb}$ and shear modulus $G_{fb}$. The dense packing limit for fibers requires $f_f < 0.83$ (Torquato, 2001). In general, five moduli are needed to describe the linear elastic response of a transversely isotropic material. However, since both the homogenized matrix and the fibers are taken to be isotropic, and since the laminate is transversely isotropic, only three of the effective moduli are independent (Hill, 1964). In the following, we present bounds and estimates for: (1) the transverse shear modulus, $G^T$, describing resistance to shearing in a plane perpendicular to the fiber axes; (2) the transverse bulk modulus, $K^T$, defined in such a way that the elastic modulus $E^T$ for stretching perpendicular to the direction of the fibers is $E^T = 9K^T G^T / (3K^T + G^T)$; and (3) the longitudinal–transverse shear modulus $G^{LT}$ describing resistance to shearing in a plane containing the fiber axes. The remaining two moduli enumerated below are dependent upon these first three: (1) the longitudinal elastic modulus $E^L$ describing resistance to stretching parallel to the fibers, and (2) the longitudinal-transverse Poisson’s ratio $v^{LT}$ that dictates the ratio between tensile strain resulting from uniaxial stretching parallel to the fibers and the associated compressive strain perpendicular to the fibers.

3.2.1. Transverse shear modulus

The Silnutzer three-point lower bound on $G^T_k$ is given by Eqs. (14) and (15) (Silnutzer, 1972, reported in Torquato, 2001) with $\phi_1$ replaced with $(f_f + f_s)$, $\phi_2$ replaced with $f_s$, $\eta_2 = 0.276f_s$, $\zeta_2 = 0.691f_s + 0.0428f^2_s$, and (i) replaced with (j), where, for example, $(G_f) = f_f G_{fb} + f_s G_{pm}$.

The Gibiansky and Torquato (1995) upper bound for $G^T_k$ is tighter than the Silnutzer bound. Making the above substitutions, the Gibiansky–Torquato upper bound is obtained from Eq. (14) using the following definition for $\Theta$:

$$\Theta^{-1} = \begin{cases} 
\frac{1}{2(G^{-1})_k} + \frac{1}{K_{max}}, & \frac{1}{K_{max}} - \frac{H}{K_{max}} \leq \frac{1}{\theta} \\
2\left(\frac{G^{-1}}{G_{fb} + K^{-1}}\right)_k - \frac{H}{G_{fb} + K^{-1}} & \frac{1}{G_{fb} + K^{-1}} \leq \frac{1}{\theta} \leq \frac{1}{\theta_{fib}},
\end{cases}$$

where $K_{max}$ is the greater of $K_{pm}$ and $K_{fb}$, $G_{fib} > G_{pm}$, $\theta = ((G^{-1})_k - 2(H/Z)(K^{-1})_k)/(1 + (H/Z))$, $H = \sqrt{\eta_1 \eta_2 G^2_{pm} - (G^2_{pm} - G^2_{fib})^2}$, and $Z = \sqrt{\frac{1}{\eta_1} + \frac{1}{\eta_2}}$.

An estimate for $G^T_k$ within these bounds is provided by Eqs. (18) and (19), with $d=2$.

3.2.2. Transverse bulk modulus

The Silnutzer three-point lower bound on $K^T_k$ is given by Eq. (17) with, as above, $\phi_1$ replaced with $(f_f + f_s)$, $\phi_2$ replaced with $f_s$, $\eta_2 = 0.276f_s$, $\zeta_2 = 0.691f_s + 0.0428f^2_s$, and (i) replaced with (j) (Torquato, 2001).

A tighter upper bound is the Gibiansky–Torquato upper bound (Gibiansky and Torquato, 1995; Torquato, 2001):

$$K^T_k \leq (K^T_k)_{ib} = \frac{(f_f + f_s)i_s(K^T(fb) - K_{pm})^2}{(K^T_k)_{fib} + \frac{G_{pm}}{G_{fb} + K_{max}}{\overline{G}_{pm}}_{\overline{G}_{fb}} + \frac{G^2_{pm} - G^2_{fib}}{K_{max}}}.$$ 

An estimate for $K^T_k$ is obtained by Eqs. (18) and (19), with $d=2$.

3.2.3. Longitudinal–transverse shear modulus

The shear modulus $G^{LT}_k$ must lie within the Hashin and Rosen (1964) bounds

$$\frac{(G^T_k + G_{fb})}{(G^T_k)_{fib} + G_{pm}} \leq G^{LT}_k \leq \frac{(G^T_k + G_{pm})}{(G^T_k)_{fib} + G_{fb}}.$$ 

The Halpin and Tsai (1967) semi-empirical equations provide estimates within these bounds for values of the parameter $\zeta$ in the range $0 \leq \zeta \leq 25$:

$$G^{LT}_k = \frac{1 + \zeta f_s}{1 - \zeta f_s} G_{pm}.$$ 

where $\zeta = \left[\frac{G_{pm}}{G_{fb}} - 1\right] / \left[G_{fb} / G_{pm} + \zeta\right]$. The value $\zeta = 2$ has been shown to provide a good estimate for laminae containing regularly spaced, aligned fibers, and will be adopted in the following (Jones, 1975).
3.2.4. Longitudinal elastic modulus

For a transversely isotropic two-phase lamina, \( E^L_k \) is dictated by the bounds or estimates of the aforementioned properties (Hill, 1964; Torquato, 2001):

\[
E^L_k = \langle E \rangle \frac{4(\nu_{pm} - \nu_i)^2}{\frac{1}{k_i} - \frac{1}{\nu_{pm} - \nu_i} + \frac{1}{\nu_{pm} + \nu_i}} \left( \frac{1}{K^l_i} + \frac{1}{G^l_i} + \frac{1}{3} \right),
\]

where \( \nu \) is Poisson’s ratio of the isotropic particulate-reinforced matrix (pm) or fibers (f), and \( k_i = K_i + G_i/3 \) for each phase \( i \).

3.2.5. Lateral/transverse Poisson’s ratio

The effective Poisson’s ratio \( \nu^T_k \) for contraction in the transverse plane associated with stretching parallel to the fibers is similarly dictated by the other material constants of the laminate (Hill, 1964; Torquato, 2001):

\[
\nu^T_k = \langle \nu \rangle \frac{(\nu_{pm} - \nu_i)^2}{\frac{1}{k_i} - \frac{1}{\nu_{pm} - \nu_i} + \frac{1}{\nu_{pm} + \nu_i}} \left( \frac{1}{K^l_i} + \frac{1}{G^l_i} + \frac{1}{3} \right),
\]

4. Framework to assess tailoring of fiber-reinforced laminates with spherical particles

We apply the Ashby (2005) method to determine through material selection charts the grading of glass spheres within the matrix of a fiberglass laminate that optimizes the mechanical response to static, instantaneous, and blast loading with a minimum of additional weight. Desirable features to enhance the response to an applied surface pressure are reductions in deflection and stress, and increases in structural and material strength. We focus on choosing the volume fraction of spheres in each lamina of a symmetric, cross-ply laminated plate that optimizes weight and stiffness in response to these loading conditions. We first assess the benefits of tailoring in a square, statically- or instantaneously loaded plate by considering the effects of all possible spatial distributions of particles in laminates containing prescribed volume fractions of fibers. An objective function indicates which of these tailorings improve upon the stiffness to weight ratio of these laminates. We then assess the efficacy of tailoring in a specific blast-loaded panel. We restrict our attention to cases in which the number of laminae \( N \) in the plate or sandwich panel is even, and all laminae are of the same thickness \( h_k \). Both representative examples illustrate the advantages of adding a relatively small amount of spherical particles to the matrix of fiber-reinforced composites.

We consider a square, simply supported cross-ply plate of side dimension \( a \), thickness \( h \), and density \( \rho \). The mass \( m \) of such a panel is

\[
m = \sum_{k=1}^{N} \rho_k h_k a^2 = \frac{2a^2h}{N} \sum_{k=1}^{N} \rho_k = \frac{9}{4} \rho \sum_{k=1}^{N} \rho_k,
\]

in which \( \rho_k \) is the density of the \( k \)th lamina in the laminate, given as a volume-weighted average of the densities \( \rho_{matrix} \), \( \rho_{spheres} \), and \( \rho_{fibers} \) of the matrix, spheres, and fibers in the lamina:

\[
\rho_k = f_1(k) \rho_{matrix} + f_2(k) \rho_{spheres} + f_3(k) \rho_{fibers},
\]

and \( \rho \) is the mean density of the plate. For the S-glass fiberglass considered in the following examples, \( \rho_{matrix} = 1500 \text{ kg/m}^3 \) and \( \rho_{spheres} = \rho_{fibers} = 2500 \text{ kg/m}^3 \).

4.1. Static and instantaneous loading

The exact solution for the static peak deflection \( w(x,y,t) \) due to an applied uniform pressure \( p_0 \) on one face of an undamped anisotropic plate is given by Lekhnitskii (1968), and the solution for an instantaneously applied dynamic loading by Soedel (2004). As described in Appendix B, we base design computations on the peak deflections resulting from a one-term approximation to these solutions. The approximation is obtained in terms of \( x \) and \( y \) coordinates in the plane of the plate, originating from one corner of the plate, by assuming a displacement field of the form \( w(x,y,t) = \beta(t) W_{11}(x,y) \), in which \( t \) is time and \( W_{11}(x,y) = \sin(\pi x/a)\sin(\pi y/a) \) represents the first mode of deformation. For a slender, isotropic plate, the accuracy of this approximation when compared to the exact solution is within 2.5%; for plates of the anisotropy range studied in the following examples, the accuracy is within 6%. The peak deflections of the plate subjected to a static or instantaneous load occur at the center of the plate and can be written (Appendix B)

\[
w_{\text{static}}^{\text{max}} \approx \left[ \frac{16p_0 a^4}{\pi^6} \right] (D_{11} + 2D_{12} + 2D_{16} + D_{22})^{-1} \quad \text{and} \quad w_{\text{instantaneous}}^{\text{max}} = 2w_{\text{static}}^{\text{max}},
\]

respectively. The composite plate stiffnesses \( D_{ij} \) of a symmetric laminate containing an even number \( N \) laminae, each of thickness \( h_k = h/N \), are
The reduced stiffnesses can be written as follows for the \( k \)-th lamina (e.g. \( J_{\text{o}}ns, 1999 \)):

\[
\begin{bmatrix}
 Q_{11} & Q_{12} & 0 \\
 Q_{12} & Q_{22} & 0 \\
 0 & 0 & Q_{44}
\end{bmatrix}^{(k)} = \begin{bmatrix}
 Z_{k\epsilon}^E + (1 - Z_k)E^T & v^{ij}E^T & 0 \\
 1 - (v^{ij})^2(E/E) & Z_{k\epsilon}^E + (1 - Z_k)E^T & 0 \\
 0 & 0 & C^{ij}
\end{bmatrix}^{(k)},
\]

where \( Z_k = 1 \) for 0° laminae oriented with the fibers parallel to the 1-direction and \( Z_k = 0 \) for 90° laminae oriented with fibers perpendicular to the 1-direction.

Eq. (28) can be solved for the panel thickness, \( h \):

\[
h^3 \approx \beta_{\text{max}} \left( \frac{8p_0\sigma^0N}{W_{\text{max}}} \right) \left( \sum_{k=1}^{N/2} Q_{ij} \right) + 2Q_{12} + 2Q_{44} + \left( k^2 - k + \frac{1}{3} \right) = \frac{1}{W_{\text{max}}} \left( \frac{8p_0\sigma^0N}{W_{\text{max}}} \right) \left( \frac{1}{\beta_{\text{max}}} \right),
\]

where \( \beta_{\text{max}} \) can be interpreted as an effective stiffness, \( \beta_{\text{max}} = 1 \) for static loading of the plate, and \( \beta_{\text{max}} = 2 \) for instantaneous loading of the plate. Note that because of the symmetricities of the problem considered, neither \( Q_{12} \) nor \( Q_{44} \), nor the sum \( (Q_{11}^{(k)} + Q_{22}^{(k)}) \) vary depending on whether a lamina is in the 0° or 90° orientation. Therefore, \( \beta_{\text{max}} \) is independent of the stacking chosen (e.g., a [90/0/90/0]s laminate will have the same value of \( \beta_{\text{max}} \) as a [90/0/90/0]s laminate).

The performance index for optimization can be identified by combining and substituting back into (26):

\[
m = 2h N^2 \sum_{k=1}^{N/2} \rho_k \approx a^2 (8\beta_{\text{max}}q)^{1/3} \left( \frac{p_0}{W_{\text{max}}} \right) \left( N\hat{\rho}\Phi^{-1/3} \right).
\]

The goal is to find a panel of minimum mass \( m \) that can provide a certain stiffness (\( p_0W_{\text{max}} \)). While the number of laminae \( N \) is held constant, the thicknesses of the laminae are allowed to vary freely to reach this optimum. Under these constraints, the only free variables on the right hand side of (32) are the density and stiffness; the panel that provides the optimal stiffness per unit weight is that which maximizes the material performance index, \( M = \Phi^{1/3}/\hat{\rho} \). As will be described in Section 5, the optimum degree of tailoring is found graphically through an Ashby-type material selection chart, a plot of \( \Phi \) against \( \Phi \) for panels containing a prescribed volume fraction of fibers and all possible distributions of spherical particles. The Ashby “selection lines” are isolines of the material performance index, \( M \).

4.2. Blast loading

We consider the response of a specific plate to an explosive blast (Houlston et al., 1985), Gupta (1985), Gupta et al. (1987), Birman and Bert (1987), Librescu and Nosier (1990a), Librescu and Na (1998a,b)). Here, we modify the approximate solution (28) to predict the response of the panel to a blast overpressure, uniformly distributed over the surface of the panel but varying with time according to the Friedlander equation (e.g., Birman and Bert, 1987)

\[
p(t) = p_0 (1 - (t/t_p)) \exp(-At/t_p),
\]

where \( p_0 \) is the peak pressure, \( t_p \) is a positive phase duration of the pulse and \( A \) is an empirical decay parameter. Reasonable values for the parameters are \( t_p = 0.1 \) s and \( A = 2 \) (Librescu and Nosier, 1990b).

The peak deflection can be estimated as (Appendix B)

\[
W_{\text{max}}(t) = W_{\text{max}}^{\text{static}} \left( \frac{\omega_1^2T_p^2}{A^2 + \omega_1^2T_p^2} \right) W(t),
\]

where

\[
W(t) = \frac{p(t)}{p_0} - \left( \frac{2A}{A^2 + \omega_1^2T_p^2} \right) \exp(-At/t_p) - \left( 1 - \frac{2A}{A^2 + \omega_1^2T_p^2} \right) \cos \omega_1 t + \left( A - \frac{A^2 - \omega_1^2T_p^2}{A^2 + \omega_1^2T_p^2} \right) \frac{\sin \omega_1 t}{\omega_1 T_p}
\]

and the fundamental frequency \( \omega_1 \) is

\[
\omega_1^2 = \frac{\pi^4}{4h^4} [D_{11} + 2(D_{12} + 2D_{66}) + D_{22}].
\]

As justified below (see the results associated with Fig. 7), the peak displacement for the example studied in Section 5 could be approximated as

\[
W_{\text{max}}^{\text{blast}} \approx W_{\text{max}}^{\text{static}} \left( 2 - \frac{\pi}{\omega_1 T_p(1 + A)} \right) = \frac{16p_0}{\pi^2\hat{\rho}c\omega_1^2} \left( 2 - \frac{\pi}{\omega_1 T_p(1 + A)} \right).
\]
Note that the first term in the parentheses in Eq. (37) is the same as Eq. (28) for an instantaneously applied load \( p_o \); the second term reflects the effect of reaching the peak pressure \( p_o \) gradually over the duration of a blast loading.

5. Numerical results

Material constants of an isotropic epoxy matrix (\( E_1 = 3.12 \) GPa and \( v_1 = 0.38 \)) containing spherical isotropic glass inclusions (\( E_2 = 76.0 \) GPa and \( v_2 = 0.25 \)) were evaluated using the two estimates and compared to the Voigt and Reuss, Hashin–Shtrikman, and three-point bounds (Fig. 1). The three-point estimate lies within all bounds considered; the Benveniste estimate (shown above to be analogous to the Kanaun–Jeulin estimate for the case of a two-phase material) falls just below the three-point bounds and coincides with the Hashin–Shtrikman lower bound. Note that the associated predictions and estimates for the Poisson ratio were inverted: the Benveniste approach predicted slightly high values. While Benveniste predictions remain within the Hashin–Shtrikman bounds, they were slightly outside the three-point bounds, particularly at larger particle volume fractions, though the deviation remained quite small. Like the Mori–Tanaka theory, the Benveniste approach is acceptable only for relatively low particle volume fractions. As will be shown below, the Kanaun–Jeulin approach for a three-phase composite has these same ranges of applicability.

For a particulate-matrix, fiber-reinforced material (glass fibers and particles, with identical moduli, and epoxy matrix), the bounds evaluated (Fig. 2) correspond to the upper and lower three-point bounds from Fig. 1, combined with the bounds presented in Section 3.2. The “two-step estimates” use the three point estimates for the particulate-reinforced matrix in conjunction with the estimates for fiber-reinforced composites presented in Section 3.2. The Kanaun–Jeulin estimate derives directly from Section 2. All moduli in these graphs were normalized by those of the epoxy matrix.

In a parameter study assessing the effects of varying volume fractions of spheres at prescribed volume fractions of fibers, the Kanaun–Jeulin estimate lies just outside of the three-point bounds, and is usually a reasonable approximation provided that the particle and fiber volume fractions are small. In all cases, the Kanaun–Jeulin estimate lies closest to the bounds for lowest volume fractions of inclusions (Fig. 2). The Kanaun–Jeulin estimate follows the lower Hashin–Shtrikman bounds (not shown) for the transverse elastic modulus (\( E_T \)) and both shear moduli (\( G \) and \( G_L \)), and lies just above the upper bound for the longitudinal elastic modulus (\( E_L \)). The “two-step” estimate lies within the three-point bounds. Estimates for Poisson’s ratio obtained using both methods are close. When the two laminae are combined into a symmetric \( 0/90/0 \) composite, the errors cancel and the Kanaun–Jeulin estimate provides estimates of some of the average reduced stiffness terms \( \overline{Q}_{ij} = (2/h) \sum_{k=1}^{N} \overline{Q}_{ik}^{k} \) (Eq. (30)) that are far improved (Fig. 3). Plotted in Fig. 3 are the sum \( \overline{Q}_{11} + \overline{Q}_{22} \) and the term \( \overline{Q}_{12} \), relevant to calculation of the bending stiffnesses \( D_{ij} \). Note that the third constant, \( \overline{Q}_{66} \), is equal to \( G_L \).
Functional grading for optimization of the stiffness to weight ratio was studied for a symmetric, 8-ply laminate consisting of epoxy, varying volume fractions $f_3$ of fibers, and all possible volume fractions $f_2$ of spherical reinforcing particles (ranging from $f_2 = 0$ to the dense packing limit, $f_2 = 0.63(1 - f_3)$; for example, for $f_3 = 0.5$, the average density $\rho$ (Eq. (27)) ranges from 2000 kg/m$^3$ (no particles) to 2315 kg/m$^3$ (dense packing limit)). The set of all possible “stiffnesses” $U$ for a laminate containing 50% by volume fibers and varying volume fractions of spherical particles, tailored with all possible ply-by-ply spatial distributions, is shown as a function of average density in Fig. 4 for four sets of mechanical properties: the upper and lower bounds from Fig. 2, the Kanaun–Jeulin estimate, and the two-step estimate. The shapes of the regions corresponding to each set of mechanical properties were nearly identical, other than an offset along the vertical axis. The scallops on the top of each region correspond to progressive addition of spherical reinforcement to the outermost plies: the leftmost scallop corresponds to a tailoring involving addition of spherical reinforcement to the outermost pair of plies, up to the maximum; the next scallop corresponds to addition of spherical reinforcement to the next pair of plies.

How much stiffening is too much? The selection lines are the isoclines of the performance index $M = \frac{U_1}{\rho} = \frac{f_3}{C_{22}}q$. The stiffness $U$ continues to increase as spherical particles are added to the inner two pairs of plies, but such configurations have a less optimal stiffness to mass ratio: the best solutions are those intersected by the highest optimization isoclines. Optimization isoclines (dashed lines, Fig. 4) appear straight because both axes of the plot are logarithmic. All $(\rho, \Phi)$ points above the
isocline that intersects the point corresponding to no reinforcement of spherical particles (lower left corner of the region, circled in Fig. 4) represent improvements in mechanical response through addition of spherical particles. The grading at the optimum (circled in Fig. 4) had no spherical particles in the inner four laminae, and the maximum possible volume fraction ($f_2 = 31.5\%$) in the outer four laminae. One of the worst possible gradings of spherical particles (also circled in Fig. 4) had the maximum possible volume fraction of spherical particles in the innermost pair of laminae, 70% of this volume fraction in the next pair, and no spherical particles in the outermost two pairs.

Note that the region corresponding to the Kanaun–Jeulin estimate obscures the region corresponding to the lower bound except at the highest volume fractions of spherical reinforcement (highest average density); at these higher values, the solution is least reliable, and the stiffness $\Phi$ can be seen to drop beneath the lower bound.

When varying volume fractions of fibers are considered (Fig. 5), the domains of possible $(p, \Phi)$ pairs for each volume fraction of fibers has approximately the same shape as those in Fig. 4, except that the region appears to rotate counterclockwise relative to the optimization isolines for higher volume fractions of fibers. The result is that the optimal tailoring of the volume fraction of spheres changes at the highest volume fraction fibers considered, so that adding the greatest possible volume of spheres to the outermost three pairs of laminae (rather than the outermost two pairs) becomes optimal. The details of the least optimal tailoring changes with fiber volume fraction, but in each case involve higher volume fractions of spheres close to the neutral axis of the laminate.

To assess the role of modulus tailoring in blast-loading, a square, symmetric, 8-ply cross-ply laminate of thickness $h = 0.008$ m and in-plane dimension $a = 0.5$ m was considered. For such a laminate, the natural frequency $\omega_1$ varied from approximately 250 Hz to 800 Hz (Fig. 6). For this natural frequency range, and using the blast considered by Librescu and
Nosier (1990a,b) \( t_p = 0.1 \) s, \( K = 2 \), several oscillations occur in the panel during the positive pressure phase of the blast duration, \( t_p \) (Fig. 7). The first and third terms in Eq. (35) are dominant over this time range. Using these two terms, \( W(t) \) is approximately (dashed lines, Fig. 7)

\[
W(t) \approx \frac{p(t)}{p_0} - \cos \omega_1 t = (1 - (t/t_p)) \exp(-At/t_p) - \cos \omega_1 t
\]

and the peak displacement occurs near \( \omega_1 t = \pi \) so that

\[
W_{\max} \approx \left( 1 - \frac{\pi}{\omega_1 t_p} \right) \exp\left( -\frac{A\pi}{\omega_1 t_p} \right) + 1 \approx 2 - \frac{\pi}{\omega_1 t_p} (1 + A).
\]

This approximation to \( W_{\max} \) is accurate to within a few percent (circles, Fig. 7). Substitution of this into Eq. (34) yielded the approximation (37). For the specific panel studied here, nearly all additions of spherical tailoring improved the stiffness per unit mass ratio of the plate (Fig. 8).

Fig. 5. Log–log Ashby-type materials selection chart, with selection lines (isoclines of the material performance index \( M = \Phi_t / \rho \)) to identify the optimal tailoring of symmetric 8-ply \( 0^\circ/90^\circ \) cross-ply laminates (volume fraction \( f_3 \) of glass fibers) using spherical glass inclusions (volume fraction \( f_2 \) allowed to vary from lamina to lamina). Improved performance is achieved for laminates tailored to have higher values of \( M \); these lie along higher isoclines, as shown. Certain distributions of spherical particles can be more effective in improving the stiffness-to-weight ratio than additional fibers.

Fig. 6. The fundamental frequency \( \omega_1 \) occurs over the range of 255–800 Hz for all possible composites and tailoring of a symmetric, 8-ply \( 0^\circ/90^\circ \) composite of the dimensions shown.
6. Discussion

This article demonstrated how tailored distributions of particle reinforcement can improve the structural response of laminated plates, using fiberglass cross-ply panels as an illustration. A series of micromechanical models were considered to obtain the results in this paper. The Kanaun–Jeulin approach suffered from limitations identical to those of the Mori–Tanaka estimate (Figs. 1 and 2). As expected from the Kanaun–Jeulin approach’s requirement that inclusion classes have minimal mechanical interactions, the estimates were best for the lowest concentrations of inclusions (Fig. 2). Errors cancelled out for the case of a cross-ply laminate (Fig. 3). At moderate concentrations of inclusions, the Kanaun–Jeulin approach offers a simple and compact approach for estimating the time-dependent response of a plate (Fig. 6). However, this approach can provide estimates that fall outside of rigorous bounds on mechanical response (Fig. 2), and must therefore be used with care.

For the example of a static or instantaneously applied pressure loading applied to a cross-ply panel, the optimization of the distribution of reinforcing particles proceeded via the Ashby approach, showing that a broad class of particle distributions led to improved stiffness to weight ratios of the panels considered (Fig. 4). As expected, the optimal tailoring involved including spherical reinforcement in the outer plies, and excluding the reinforcement from the inner plies. Poorly-chosen tailoring of spherical reinforcement particles leads to a reduction in the stiffness to weight ratio of the panels. However, well-chosen spherical reinforcement, added to the outermost plies, was more effective than the uniform addition of fibers to all laminae (Fig. 5).

![Fig. 7.](image1)

For the blast and plate considered, the peak displacement increased as the fundamental frequency, \(\omega_p\), increased. The time and magnitude of this peak displacement was approximated to within a few percent by Eq. (37).

![Fig. 8.](image2)

For the specific symmetric 8-ply \(0^\circ/90^\circ\) cross-ply laminate plate considered, nearly all increases in mass resulting from the addition of spherical particles leads to an improvement in the stiffness-to-weight ratio. The only exceptions are cases in which particles are added only to the innermost laminae in the composite. Both axes are linear.
For blast loading, a closed form estimate was developed to predict the peak displacement of a panel in response to a Friedlander pressure pulse. A blast-resistant panel can withstand only a limited deflection before losing strength; while other aspects of blast resistance such as improved thermal resistivity from glass particles and the ability to tailor stiffness in the vicinity of stress concentrations are certainly important (e.g. Budiansky et al., 1993; Genin and Hutchinson, 1999), the focus here was exclusively on the improvement of stiffness to weight ratio. For the specific panel and blast chosen for detailed study, the peak displacement occurred at a well-defined moment shortly after the arrival of the blast “overpressure” wave. This time was a function of the fundamental frequency; the greater the fundamental frequency, the sooner after the blast and greater in magnitude was the peak displacement (Fig. 7). However, the magnitude of this peak decreased with both increasing stiffness and decreasing mass, and hence with increasing particle content in the laminate. The improvements from stiffness increases dominated over the deleterious effects of increased fundamental frequency, and the stiffness to weight ratio for the blast-loaded panel improved for nearly all additions of glass particles, no matter how poorly the tailoring was chosen (Fig. 8).

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Appendix A. Eshelby tensors

Eshelby’s tensor for a spherical inclusion embedded within an isotropic matrix are the following (Eshelby, 1959; Torquato, 2001):

\[
S_{ijkl} = \frac{5v_1 - 1}{15(1 - v_1)} \delta_{ij} \delta_{kl} + \frac{4 - 5v_1}{15(1 - v_1)} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),
\]

(A1)

where \(v_1\) Poisson’s ratio for the matrix and \(\delta_i\) is Kronecker’s delta. For needle-shaped inclusions, with the axis of the needles aligned in the 1-direction, the non-zero components are

\[
\begin{align*}
S_{1112} &= S_{3333} = \frac{5 - 4v_1}{8(1 - v_1)}, \\
S_{1111} &= S_{3311} = \frac{v_1}{2(1 - v_1)}, \\
S_{1112} &= S_{3313} = \frac{3 - 4v_1}{8(1 - v_1)}, \\
S_{1112} &= S_{3313} = 1/4.
\end{align*}
\]

(A2)

Appendix B. Approximate response of a simply supported orthotropic plate to an arbitrary dynamic pressure loading

The Love equation for the displacement field \(w(x, y, t)\) of an undamped, square plate lying in the x-y plane that is subjected to a uniform, time-varying pressure \(p(t)\) is (e.g. Soedel, 2004)

\[
L\{w(x, y, t)\} + \rho h \frac{d^2 w(x, y, t)}{dt^2} = p(t),
\]

(B1)

where \(\rho\) is the mean density of the plate, \(h\) is the thickness of the plate, and the operator \(L\{w(x, y, t)\}\) can be written as

\[
L\{w(x, y, t)\} = D_{11} \frac{\partial^4 w(x, y, t)}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w(x, y, t)}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w(x, y, t)}{\partial y^4}.
\]

(B2)

The exact solution to this problem can be written in terms of an infinite series using the standard Fourier decomposition (e.g. Soedel, 2004). We focus attention on an approximate solution incorporating only the first mode, \(W_{11}(x, y)\):

\[
w(x, y, t) \approx \beta_1(t) W_{11}(x, y) = \beta_1(t) \sin \frac{\pi x}{a} \sin \frac{\pi y}{a},
\]

(B3)

where the modal participation factor \(\beta_1(t)\) is found from the following convolution integral (Soedel, 2004):

\[
\beta_1(t) = \frac{1}{\omega_1} \int_0^t A_1(\tau) \sin \omega_1(t - \tau) d\tau,
\]

(B4)
in which the fundamental frequency \(\omega_1\) can be written as in Eq. (35), and the composite plate stiffnesses \(D_1\) are defined in Eq. (29). The amplitude \(A_1(t)\) in the convolution integral, with units of acceleration, can be written
where \( \Omega \) represents the surface of the plate. Thus,
\[
w(x, y, t) \approx \frac{16}{\pi^2 \rho h \omega_1} \int_0^t p(\tau) \sin \omega_1 (t - \tau) d\tau
\]
and
\[
w_{\text{max}}(t) = w(a, a, t) \approx \frac{16}{\pi^2 \rho h \omega_1} \int_0^t p(\tau) \sin \omega_1 (t - \tau) d\tau.
\]
For the case of an instantaneously applied loading \( p_0 \) at time \( t = 0 \), (B7) yields
\[
w_{\text{max}}(t) = \frac{16p_0}{\pi^2 \rho h \omega_1} (1 - \cos \omega_1 t) = w_{\text{max}}^{\text{static}} (1 - \cos \omega_1 t),
\]
where \( w_{\text{max}}^{\text{static}} = 16p_0/\pi^2 \rho h \omega_1^2 \) is the peak deflection that would result from the quasi-static application of a pressure \( p_0 \) to the face of the plate.

For a blast overpressure, uniformly distributed over the surface of the plate but varying with time according to the Friedlander equation (Eq. (33)), Eq. (B7) yields
\[
w_{\text{max}}(t) = w_{\text{max}}^{\text{static}} \left( \frac{\omega_1^2 \rho^2}{A^2 + \omega_1^2 \rho^2} \right) W(t),
\]
where
\[
W(t) = \frac{p(t)}{p_0} - \frac{2A}{A^2 + \omega_1^2 \rho^2} \exp(-At/t_\rho) - \left( 1 - \frac{2A}{A^2 + \omega_1^2 \rho^2} \right) \frac{\cos \omega_1 t}{A^2 + \omega_1^2 \rho^2} \sin \omega_1 t.
\]

References


