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A THEORY OF THERMOVISCOELASTIC DIELECTRICS

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We study finite deformations of heat conducting viscous dielectric solids and derive constitutive relations that satisfy an entropy inequality. These are simplified for orthotropic, transversely isotropic, and isotropic materials. For each class of materials, linear constitutive relations valid for infinitesimal deformations are derived.

Bell [1] documented that the response of all materials is, in general, nonlinear. For piezoelectric ceramics, Lazarus and Crawley [2, 3] stated that the material parameters should be taken to depend upon the induced strains. Norwood et al. [3] and Kulkarni and Hanagud [4] used a Neo-Hookean constitutive relation to model the response of piezoelectric ceramics; the Neo-Hookean model is the simplest model for a nonlinear elastic material. A finite deformation theory of elastic dielectrics was developed, amongst others, by Toupin [5] and Eringen and Maugin [6]. Tiersten [7] used the theory of invariants [8] to derive electroelastic equations that are cubic in the mechanical displacement gradient and the electric potential gradient.

Transversely isotropic piezoelectric materials are widely used in transducers and smart structures. In the latter application it is desirable that these materials exhibit damping, and accordingly these are now being manufactured from composites that show viscous effects. Our goal here is to develop a finite deformation theory for nonmagnetizable but heat-conducting unconstrained dielectrics and to use the theory of invariants to derive the simplest form of constitutive relations for orthotropic, transversely isotropic, and isotropic dielectrics.

BALANCE LAWS

We consider a nonmagnetizable dielectric solid with an infinitesimal memory of mechanical deformations roughly equivalent to that of a Navier–Stokes fluid; the material studied here is simple in the sense of Noll (cf. Truesdell and Noll [9]), is
heat conducting, and exhibits piezoelectric and viscous effects. In spatial description and with quasi-electrostatic approximation, the balance of mass, linear momentum, moment of momentum, internal energy, and Maxwell equations are (Eringen and Maugin [6])

\[
\begin{align*}
\dot{\rho} + \rho \text{div}\, v &= 0 \\
\rho \dot{v} &= \text{div}\, \dot{t} + (q_e - \text{div}\, p)e + \rho \mathbf{b} \\
t_\mathbf{a} &= - (p \otimes e)_\mathbf{a} \\
\rho \dot{\phi} &= \text{tr}(tL) + \text{div}\, q - \mathbf{p} \cdot \dot{e} - \rho h \\
\text{div}\, \mathbf{d} &= q_e \\
\text{curl}\, e &= 0
\end{align*}
\] (1)

Here \( \rho \) is the present mass density of a material particle, \( v \) is its velocity, \( d \) is the dielectric displacement vector, \( p \) is the polarization vector, \( e \) is the electric field, \( b \) is the body force per unit mass, \( q_e \) is the prescribed body charge, \( t \) is the total Cauchy stress given by

\[
t = \dot{t} - \mathbf{p} \otimes e
\] (2a)

where \( a \otimes b \) denotes the tensor product between vectors \( a \) and \( b \), \( t_\mathbf{a} \) is the antisymmetric part of \( t \), and

\[
(a \otimes b)_\mathbf{a} = a \otimes b - b \otimes a
\] (2b)

Note that \( \dot{t} \) is symmetric but \( t \) is not. In Eq. (1)_4, \( \phi \) is the modified specific internal energy, \( L = \text{grad}\, v \) is the velocity gradient, \( q \) is the heat flux per unit present area, and \( h \) is the source of internal energy. Vectors \( p, e, \) and \( d \) are related by

\[
d = e + p
\] (3)

The balance laws (1) are supplemented by an entropy inequality, which we take as the Clausius-Duhem inequality (cf. Truesdell and Noll [9]):

\[
\rho \gamma = \rho \dot{\gamma} - \text{div}(q/\theta) - \frac{\rho h}{\theta} \geq 0
\] (4)

Here \( \gamma \) is the specific entropy, \( \theta > 0 \) is the absolute temperature of a material point, and \( \gamma \) is the specific production of entropy. In terms of the free energy density

\[
\psi = \phi - \eta \theta
\] (5)

we obtain from Eqs. (1)_4 and (4)

\[
- \rho(\dot{\psi} + \eta \dot{\theta}) + \text{tr}(tL) - \frac{1}{\theta}q \cdot \text{grad}\, \theta - \mathbf{p} \cdot \dot{e} \geq 0
\] (6)

as the reduced entropy inequality; here we have eliminated \( h \) from (1)_4 and (4).
Whereas balance laws (1) are valid for all materials and processes, only those processes that satisfy the entropy inequality (6) are admissible in a real material. Here we adopt the viewpoint that the balance of moment of momentum (11) is identically satisfied and the entropy inequality (6) imposes restrictions on the admissible constitutive relations.

CONSTITUTIVE RELATIONS

Anisotropic Materials

Constitutive relations distinguish one material from another. Here we consider a class of solids that exhibit very short memory of the mechanical deformations essentially equivalent to that of Navier-Stokes fluids. For solids the material response also depends upon the choice of the reference configuration, which henceforth is kept fixed. We use the referential description of motion and assume that the response of a material point $X$ is determined by values of

$$\Gamma = \{S, \dot{S}, \theta, G, E\}$$

where

$$2S = F^{T}F - 1 \quad G = \dot{\theta}/\partial X$$

$$E = F^{-1}e \quad F = \partial \mathbf{x}/\partial X$$

$x = x(X, t)$ gives the present position of the material particle that occupied place $X$ in the reference configuration. Let

$$\Lambda = \{\dot{T}, \eta, P, Q, \psi\}$$

where

$$\dot{T} = JF^{-1}\dot{t}(F^{-1})^{T} \quad Q = JF^{-1}q \quad P = JF^{-1}p \quad J = \text{det}[F]$$

Then our constitutive hypothesis is that

$$\Lambda(X, t) = \mathcal{F}(\Gamma(X, t), X)$$

Henceforth we restrict ourselves to one material point and, therefore, omit the explicit dependence of $\mathcal{F}$ upon $X$.

Under a change of frame of reference both $\Gamma$ and $\Lambda$ are scalar quantities; thus Eqs. (11) are trivially frame-indifferent.

In the terms of $\Gamma$ and $\Lambda$, the reduced entropy inequality (6) becomes

$$-\rho_0(\dot{\psi} + \eta \dot{\theta}) + \text{tr}(\dot{T}\dot{S}) - \frac{1}{\theta} Q \cdot G - P \cdot E \geq 0$$
With \( \rho_0 \psi(X, t) = \Sigma(\Gamma(X, t)) \) and the following usual arguments (see, for example, Coleman and Noll [10]), we obtain

\[
\eta = -\frac{1}{\rho_0} \frac{\partial \Sigma}{\partial \theta} \quad \frac{\partial \Sigma}{\partial S} = 0 \quad P = -\frac{\partial \Sigma}{\partial E} \quad \frac{\partial \Sigma}{\partial G} = 0 \quad \hat{T}^e = \frac{\partial \Sigma}{\partial S} \tag{13}
\]

and

\[
\mathcal{R} = \text{tr}(\hat{T}S) - \frac{1}{\theta} Q \cdot G \geq 0 \tag{14}
\]

where

\[
\Sigma = \Sigma(\hat{T}) \quad \hat{T} = (S, \theta, E) \quad \hat{T} = \hat{T}^e + \hat{T}^{ne} \quad \hat{T}^{ne} = \hat{T}^{ne}(\Gamma) \tag{15}
\]

Inequality (14) implies that

\[
\hat{T}^{ne}(S, 0, \theta, 0, E) = 0 \tag{16a}
\]

\[
Q(S, 0, \theta, 0, E) = 0
\]

A process in which \( \dot{S} = 0 \) and \( G = 0 \) is referred to as an equilibrium process. Thus in an equilibrium process, \( \hat{T}^{ne} \) and \( Q \) vanish identically. Also

\[
\frac{\partial^2 \mathcal{R}}{\partial S \partial G} \bigg|_{\dot{S} = G = 0} \geq 0 \tag{16b}
\]

in the sense that its determinant and all of its subdeterminants must be nonnegative. Functions \( \Sigma(\hat{T}), \hat{T}^{ne}(\Gamma), \) and \( Q(\Gamma) \) satisfying (14) and (16) characterize an anisotropic nonmagnetic dielectric material with a short memory; constitutive relations for \( \eta, P, \) and \( \hat{T}^e \) are obtained by using (13). Depending upon the material symmetries, one can simplify these further.

### Isotropic Materials

For isotropic materials, functions \( \Sigma, \hat{T}^{ne}, \) and \( Q \) must satisfy

\[
\Sigma(\hat{T}) = \Sigma(\hat{T}) \quad \hat{T}^{ne}(\Gamma) = H \hat{T}^{ne}(T) H^T \quad Q(\Gamma) = HQ(\Gamma) \tag{17}
\]

for every orthogonal tensor \( H \) where

\[
\hat{T} = (HSH^T, \theta, HE)
\]

\[
\Gamma = (HSH^T, HSH^T, \theta, HG, HE)
\]
Representation theorems (cf. Wang [11], Smith et al. [12], and Zheng [13]) imply that

\[ \Sigma = \Sigma(I_1, I_2, \ldots, I_6, \theta) \]  

\( \hat{T}^{ne} = \alpha_0 \mathbf{1} + \alpha_1 \mathbf{S} + \alpha_2 \mathbf{\dot{S}} + \alpha_3 \mathbf{G} \otimes \mathbf{G} + \alpha_4 \mathbf{E} \otimes \mathbf{E} \)

\[ + \alpha_5 (\mathbf{E} \otimes \mathbf{G}) + \alpha_6 \mathbf{S}^2 + \alpha_7 \mathbf{\dot{S}}^2 + \alpha_8 (\mathbf{G} \otimes \mathbf{SG}) + \alpha_9 (\mathbf{G} \otimes \mathbf{S}^2 \mathbf{G}) \]

\[ + \alpha_{10} (\mathbf{E} \otimes \mathbf{SE}) + \alpha_{11} (\mathbf{E} \otimes \mathbf{S}^2 \mathbf{E}) + \alpha_{12} (\mathbf{G} \otimes \mathbf{SG}) + \alpha_{13} (\mathbf{G} \otimes \mathbf{S}^2 \mathbf{G}) \]

\[ + \alpha_{14} (\mathbf{E} \otimes \mathbf{SE}) + \alpha_{15} (\mathbf{E} \otimes \mathbf{S}^2 \mathbf{E}) + \alpha_{16} (\mathbf{SS}^2) + \alpha_{17} (\mathbf{S}^2 \mathbf{S}) + \alpha_{18} (\mathbf{S}^2 \mathbf{S}^2) \]

\[ + \alpha_{19} (\mathbf{S}(\mathbf{G} \otimes \mathbf{E})) + \alpha_{20} (\mathbf{S}(\mathbf{G} \otimes \mathbf{E})) \]

\[ = \mathbf{Q} = \beta_1 \mathbf{G} + \beta_2 \mathbf{E} + \beta_3 \mathbf{SG} + \beta_4 \mathbf{SE} + \beta_5 \mathbf{\dot{S}G} + \beta_6 \mathbf{\dot{S}E} \]

\[ + \beta_7 \mathbf{S}^2 \mathbf{G} + \beta_8 \mathbf{S}^2 \mathbf{E} + \beta_9 \mathbf{\dot{S}G} + \beta_{10} \mathbf{\dot{S}E} + \beta_{11} (\mathbf{SS}) + \beta_{12} (\mathbf{S}^2 \mathbf{S}) \mathbf{E} \]

where

\[ (\mathbf{u} \otimes \mathbf{v})_s = \mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u} \quad (\mathbf{u} \otimes \mathbf{v})_a = \mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u} \]

\[ (\mathbf{AB})_a = \mathbf{AB} - \mathbf{BA} \quad (\mathbf{AB})_s = (\mathbf{AB} + \mathbf{BA}) \]

\[ I_1 = \text{tr} \mathbf{S} \quad I_2 = \text{tr} \mathbf{S}^2 \quad I_3 = \text{tr} \mathbf{S}^3 \quad I_4 = \mathbf{E} \cdot \mathbf{E} \]

\[ I_5 = \mathbf{E} \cdot \mathbf{SE} \quad I_6 = \mathbf{E} \cdot \mathbf{S}^2 \mathbf{E} \]

Coefficients \( \alpha_0, \alpha_1, \ldots, \alpha_{20} \) and \( \beta_1, \ldots, \beta_{12} \) are functions of \( \theta \), and invariants \( I_1, I_2, \ldots, I_{28} \), where

\[ I_7 = \mathbf{G} \cdot \mathbf{G} \quad I_8 = \mathbf{G} \cdot \mathbf{E} \quad I_9 = \text{tr} \mathbf{\dot{S}} \quad I_{10} = \text{tr} \mathbf{\dot{S}^2} \quad I_{11} = \text{tr} \mathbf{\dot{S}^3} \]

\[ I_{12} = \mathbf{G} \cdot \mathbf{SG} \quad I_{13} = \mathbf{E} \cdot \mathbf{SE} \quad I_{14} = \mathbf{G} \cdot \mathbf{SG} \]

\[ I_{15} = \mathbf{G} \cdot \mathbf{SE} \quad I_{16} = \mathbf{G} \cdot \mathbf{SE} \quad I_{17} = \mathbf{G} \cdot \mathbf{S}^2 \mathbf{G} \]

\[ I_{18} = \mathbf{G} \cdot \mathbf{S}^2 \mathbf{G} \quad I_{19} = \mathbf{E} \cdot \mathbf{S}^2 \mathbf{E} \quad I_{20} = \mathbf{G} \cdot \mathbf{SS} \mathbf{G} \quad I_{21} = \mathbf{E} \cdot \mathbf{SS} \mathbf{E} \]

\[ I_{22} = \mathbf{G} \cdot \mathbf{S}^2 \mathbf{E} \quad I_{23} = \mathbf{G} \cdot \mathbf{S}^2 \mathbf{E} \quad I_{24} = \mathbf{G} \cdot \mathbf{SS} \mathbf{G} \]

\[ I_{25} = \text{tr} (\mathbf{SS}) \quad I_{26} = \text{tr} (\mathbf{S}^2 \mathbf{S}) \]
In order that Eqs. (20a, b) satisfy (16a)

\[ \bar{\alpha}_0 = \bar{\alpha}_1 = \bar{\alpha}_4 = \bar{\alpha}_6 = \bar{\alpha}_{10} = \bar{\alpha}_{11} = 0 \]

where a superimposed bar indicates their values when \( \hat{\mathbf{S}} = \mathbf{G} = 0 \).

Equations (13) and (19) yield

\[ \hat{T}' = \gamma_1 \mathbf{1} + 2 \gamma_2 \mathbf{S} + 3 \gamma_3 \mathbf{S}^2 + \gamma_4 \mathbf{E} \otimes \mathbf{E} + \gamma_5 (\mathbf{SE} \otimes \mathbf{E}) \]

\[ \mathbf{P} = -2 [ \gamma_4 \mathbf{E} + \gamma_5 \mathbf{SE} + \gamma_6 \mathbf{S}^2 \mathbf{E} ] \]

\[ \eta = \frac{1}{\rho_0} \frac{\partial \Sigma}{\partial \theta} = \hat{\eta}(I_1, \ldots, I_6, \theta) \]

where \( \gamma_i = \partial \Sigma / \partial I_i, i = 1, 2, \ldots, 6 \).

The challenging (probably insurmountable) task now is to devise experiments to determine the functional dependence of \( \Sigma \) and \( \alpha_1, \alpha_2, \ldots, \alpha_{20}; \beta_1, \ldots, \beta_{12} \) upon their arguments. Since isotropic materials have the largest symmetry group, they require the determination of a minimum number of functions.

**Transversely Isotropic Materials**

Piezoceramics are usually poled in a certain direction and can be reasonably modeled as transversely isotropic. Let the unit vector \( \mathbf{N} \) be the axis of transverse isotropy. The representation theorem of Zheng [13] yields

\[ \hat{\sigma}^m = \alpha_0 \mathbf{1} + \alpha_1 \mathbf{S} + \alpha_2 \hat{\mathbf{S}} + \alpha_3 \mathbf{S}^2 + \alpha_4 \hat{\mathbf{S}}^2 + \alpha_5 (\mathbf{S} \hat{\mathbf{S}}) \]

\[ + \alpha_6 \mathbf{E} \otimes \mathbf{E} + \alpha_7 \mathbf{G} \otimes \mathbf{G} + \alpha_8 (\mathbf{G} \otimes \mathbf{E}) + \alpha_9 \mathbf{N} \otimes \mathbf{N} \]

\[ + \alpha_{10} (\mathbf{N} \otimes \mathbf{S} \mathbf{N}) + \alpha_{11} (\mathbf{N} \otimes \mathbf{S}^2 \mathbf{N}) + \alpha_{12} (\mathbf{N} \otimes \hat{\mathbf{S}} \mathbf{N}) \]

\[ + \alpha_{13} (\mathbf{N} \otimes \hat{\mathbf{S}}^2 \mathbf{N}) + \alpha_{14} (\mathbf{N} \otimes \mathbf{E}) + \alpha_{15} (\mathbf{N} \otimes \mathbf{G}) \]

\[ + \alpha_{16} (\mathbf{S} (\mathbf{N} \otimes \mathbf{E}) \mathbf{S}) + \alpha_{17} (\mathbf{S} (\mathbf{N} \otimes \mathbf{G}) \mathbf{S}) + \alpha_{18} (\hat{\mathbf{S}} (\mathbf{N} \otimes \mathbf{E}) \hat{\mathbf{S}}) \]

\[ + \alpha_{19} (\hat{\mathbf{S}} (\mathbf{N} \otimes \mathbf{G}) \hat{\mathbf{S}}) \]

\[ \mathbf{Q} = \beta_1 \mathbf{E} + \beta_2 \mathbf{G} + \beta_3 \mathbf{S} \mathbf{E} + \beta_4 \mathbf{S} \mathbf{G} + \beta_5 \hat{\mathbf{S}} \mathbf{E} + \beta_6 \hat{\mathbf{S}} \mathbf{G} + \beta_7 \mathbf{N} \]

\[ + \beta_8 \mathbf{S} \mathbf{N} + \beta_9 \hat{\mathbf{S}} \mathbf{N} + \beta_{10} \mathbf{S}^2 \mathbf{N} + \beta_{11} \hat{\mathbf{S}}^2 \mathbf{N} + \beta_{12} (\mathbf{S} \hat{\mathbf{S}}) \mathbf{N} \]
where
\[(S(N \otimes E) \text{,}_a)_b = S(N \otimes E - E \otimes N) - (N \otimes E - E \otimes N)S\]  
(27)

\[\alpha_0, \ldots, \alpha_{19}, \beta_1, \ldots, \beta_{12} \text{ are functions of } \theta \text{ and the invariants } J_1, \ldots, J_{33}.\] These invariants are defined as:

\[J_i = I_i \quad i = 1, \ldots, 5\]
\[J_i = I_{i-2} \quad i = 11, \ldots, 18 \quad J_i = I_{i+6} \quad i = 19, 20, 21\]

\[J_6 = N \cdot SN \quad J_{22} = N \cdot \dot{S}N \quad J_7 = N \cdot \dot{S}^2N \quad J_{23} = N \cdot \ddot{S}N\]

\[J_8 = N \cdot E \quad J_{25} = N \cdot G \quad J_9 = N \cdot SE \quad J_{24} = N \cdot S\dot{S}N\]

\[J_{10} = N \cdot \dot{S}^2E \quad J_{27} = N \cdot SG \quad J_{28} = N \cdot \dot{S}E \quad J_{29} = N \cdot \ddot{S}G\]

\[J_{31} = N \cdot S^2G \quad J_{32} = N \cdot \dot{S}^2E \quad J_{33} = N \cdot \dot{S}^2G \quad J_{26} = N \cdot S\dot{SE} \quad J_{30} = N \cdot S\ddot{SE}\]

For Eqs. \(26a, b\) to satisfy Eqs. \(16a\) we must have

\[\bar{\alpha}_0 = \bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha}_6 = \bar{\alpha}_9 = \bar{\alpha}_{10} = \bar{\alpha}_{11} = \bar{\alpha}_{14} = \bar{\alpha}_{16} = 0\]

\[\bar{\beta}_1 = \bar{\beta}_3 = \bar{\beta}_7 = \bar{\beta}_8 = \bar{\beta}_{10} = 0\]  
(29)

where a superimposed bar indicates their values when \(\dot{S} = G = 0\). Relations (13), and (13), imply that

\[
\hat{T}' = \gamma_11 + 2\gamma_2S + 3\gamma_3S^2 + \gamma_5E \otimes E + \gamma_6N \otimes N + \frac{1}{2} \gamma_9(N \otimes SN)_s + \gamma_{10}(N \otimes SE)_s
\]

\[P = -[2\gamma_4E + 2\gamma_5SE + \gamma_9N + \gamma_9SE + \gamma_{10}S^2N]\]

\[\eta = -\frac{1}{\rho_0} \frac{\partial \Sigma}{\partial \theta} = \hat{\eta}(J_1, J_2, \ldots, J_{10}, \theta)\]  
(30)

where \(\gamma_i = \partial \Sigma / \partial J_i\).

**Orthotropic Materials**

An orthotropic material point has three orthogonal planes of symmetry; let \(e_1, e_2, e_3\) denote unit vectors perpendicular to these planes and

\[M = e_2 \otimes e_2 - e_3 \otimes e_3\]  
(31)

\[ \Sigma = \Sigma (I_1, \ldots, I_6, K_1, \ldots, K_7, \theta) \] (32)

\[
\hat{T}^{ae} = \alpha_0 I + \alpha_1 S + \alpha_2 \tilde{S} + \alpha_3 S^2 + \alpha_4 \tilde{S}^2 + \alpha_5 (S \tilde{S}) + \alpha_6 E \otimes E
\]
\[ + \alpha_7 G \otimes G + \alpha_8 (E \otimes SE)_a + \alpha_9 (E \otimes \tilde{S}E)_a + \alpha_{10} (G \otimes SG)_a
\]
\[ + \alpha_{11} (G \otimes \tilde{S}G)_a + \alpha_{12} (E \otimes G)_a + \alpha_{13} (S (E \otimes G)_a)_a
\]
\[ + \alpha_{14} (G \otimes \tilde{S}G)_a + \alpha_{15} M + \alpha_{16} e_1 \otimes e_i
\]
\[ + \alpha_{17} (MS)_a + \alpha_{18} (M \tilde{S})_a + \alpha_{19} (e_1 \otimes S e_1)_a + \alpha_{20} (e_1 \otimes \tilde{S} e_1)_a
\]
\[ + \alpha_{21} (E \otimes ME)_a + \alpha_{22} (G \otimes MG)_a + \alpha_{23} (E \cdot e_1)(e_1 \otimes E)_a
\]
\[ + \alpha_{24} (G \cdot e_1)(e_1 \otimes G)_a + \alpha_{25} (M (E \otimes G)_a)_a \] (33a)

\[
Q = \beta_1 E + \beta_2 G + \beta_3 SE + \beta_4 SG + \beta_5 \tilde{S}E + \beta_6 \tilde{S}G + \beta_7 S^2 E + \beta_8 S^2 G
\]
\[ + \beta_9 \tilde{S}^2 E + \beta_{10} \tilde{S}^2 G + \beta_{11} ME + \beta_{12} MG + \beta_{13} (E \cdot e_1) e_i + \beta_{14} (G \cdot e_1) e_i
\]
\[ + \beta_{15} (MS)_a E + \beta_{16} (MS)_a G + \beta_{17} (M \tilde{S})_a E + \beta_{18} (M \tilde{S})_a G \] (33b)

where \( \alpha_0, \alpha_1, \ldots, \alpha_{25}; \beta_1, \ldots, \beta_{18} \) are functions of \( \theta \) and invariants \( I_1, \ldots, I_{27}; K_1, \ldots, K_{21} \). Invariants \( K_1, \ldots, K_{21} \) are defined below.

\[
K_1 = \text{tr}(MS) \quad K_2 = \text{tr}(MS^2) \quad K_3 = e_1 \cdot Se_i
\]
\[
K_4 = e_1 \cdot S^2 e_1 \quad K_5 = E \cdot ME \quad K_6 = (E \cdot e_1)^2 \quad K_7 = E \cdot MSE
\]
\[
K_8 = \text{tr}(M \tilde{S}) \quad K_9 = \text{tr}(MS^2) \quad K_{10} = e_1 \cdot \tilde{S} e_1 \quad K_{11} = e_1 \cdot \tilde{S}^2 e_1
\]
\[
K_{12} = \text{tr}(M S \tilde{S}) \quad K_{13} = G \cdot MG \quad K_{14} = (G \cdot e_1)^2
\]
\[
K_{15} = G \cdot MSG \quad K_{16} = E \cdot M \tilde{S} E \quad K_{17} = G \cdot M \tilde{S} G \quad K_{18} = E \cdot MG
\]
\[
K_{19} = (E \cdot e_1)(G \cdot e_1) \quad K_{20} = E \cdot (MS)_a G \quad K_{21} = E \cdot (M \tilde{S})_a G
\]

Equations (33a, b) and (16) imply that

\[
\bar{\alpha}_0 = \bar{\alpha}_1 = \bar{\alpha}_3 = \bar{\alpha}_6 = \bar{\alpha}_8 = \bar{\alpha}_{15} = \bar{\alpha}_{16} = \bar{\alpha}_{17} = \bar{\alpha}_{19} = \bar{\alpha}_{21} = \bar{\alpha}_{23} = 0
\]
\[
\bar{\beta}_1 = \bar{\beta}_3 = \bar{\beta}_7 = \bar{\beta}_{11} = \bar{\beta}_{13} = \bar{\beta}_{15} = 0
\] (35)
Substitution from Eq. (32) into Eqs. (13), and (13), yields

$$P = -2 \left[ \gamma_E E + \gamma_5 SE + \gamma_6 S^2 E + \delta_5 ME + \delta_0 (E \cdot e_i) e_i + \delta_7 MSE \right]$$  \hspace{1cm} (36)

$$\hat{T}^e = \gamma_1 I + 2 \gamma_2 S + 3 \gamma_3 S^2 + \gamma_5 SE \otimes E + \gamma_6 (SE \otimes E) +$$
$$\begin{array}{c}
+ \delta_1 M + \delta_2 (MS) + \delta_3 (e_i \otimes e_i + 2 \delta_4 (e_i \otimes Se_i)) \\
+ \frac{1}{2} \delta_7 (E \otimes ME),
\end{array}$$  \hspace{1cm} (37)

$$\eta = -\frac{1}{\rho_0} \partial \Sigma / \partial \theta = \hat{\eta}(I_1, \ldots, I_6, K_1, \ldots, K_7, \theta)$$

where $y_i = \frac{\partial \Sigma}{\partial I_i}, \delta_i = \frac{\partial \Sigma}{\partial K_i}$.

**LINEAR CONSTITUTIVE RELATIONS**

Linear constitutive relations are appropriate when the magnitudes of displacement gradients and, therefore, of the infinitesimal strain tensor $\epsilon$, its time rate of change $\dot{\epsilon}$, electric field $E$, the relative temperature change $\theta = (\theta - \theta_0) / \theta_0$, and of the gradients of the temperature change are small as compared to one. Since linear theories are easy to work with, we give below linear constitutive relations for different material symmetries.

**Anisotropic Materials**

We assume that $\Sigma, \hat{T}^{ae},$ and $Q$ are smooth functions of their arguments and can be expanded in terms of a Taylor series about the values their arguments take in the reference configuration. The finite strain tensor $S$ is expressed in terms of displacement gradients. Using Eqs. (13) and, keeping terms of order one in $\epsilon, \dot{\epsilon}, \theta, \hat{G} = \text{Grad} \theta,$ and $E,$ and recalling Eq. (16), we arrive at the following linear constitutive relations:

$$\hat{T}^e = \hat{T}_0^e + \alpha \dot{\theta} + C^\epsilon \epsilon - D^e E$$

$$\hat{T}^{ae} = C^{ae} \dot{\epsilon} + \alpha'^{ae} \hat{G}$$

$$P = P_0 + \omega \dot{\theta} + A \epsilon + D^e E$$

$$Q = -K \hat{G} + K^\epsilon \dot{\epsilon}$$

$$\eta = \eta_0 + \frac{1}{\rho_0} (\mu \dot{\theta} + \mu^t \cdot \epsilon + \omega \cdot E)$$
Here $\hat{T}_0^e$, $P_0$, and $\eta_0$ are the values of the equilibrium stress $T^e$, polarization $P$, and the specific entropy $\eta$ in the reference configuration. Necessary conditions for Eqs. (38) to satisfy the reduced entropy inequality (14) imply that the fourth-order viscosity tensor $C^{ne}$ and the second-order thermal conductivity tensor $K$ are positive definite; however, $K$ need not be symmetric. The fourth-order tensor $C^e$ is called the elasticity tensor and the usual symmetry $C^e = C^{eT}$ assumed for it is an additional assumption rather than a consequence of the entropy inequality, components of the second-order tensor $\alpha^e$ are the thermal stress moduli, of the third-order tensor $D^e$ are the piezoelectric moduli, $\omega$ may be called the pyroelectric polarizability vector, and the second-order tensor $A$ is the dielectric susceptibility tensor. Tensors such as $\alpha^e$ and $C^e$ are functions of $X$ only and are constants for a homogeneous material.

**Isotropic Materials**

For isotropic materials, Eqs. (38) simplify to

$$
\hat{T}^e = \lambda_0 \mathbf{1} + \alpha \hat{\theta} \mathbf{1} + \lambda_0 (\text{tr} \mathbf{e}) \mathbf{1} + 2 \mu_0 \mathbf{e}
$$

$$
\hat{T}^{ne} = \lambda_1 (\text{tr} \mathbf{e}) \mathbf{1} + 2 \mu_1 \mathbf{e}
$$

$$
P = a \mathbf{E}
$$

$$
Q = -k \hat{G}
$$

$$
\eta = \eta_0 + \alpha \text{tr} \mathbf{e} + \beta \hat{\theta}
$$

where $\lambda_0$ and $\mu_0$ are the Lamé’s constants, $\alpha$ is the coefficient of thermal expansion, $\lambda_1$ and $\mu_1$ are the viscosities of the material, $a$ is the dielectric constant, and $k > 0$ is the thermal conductivity of the material. As expected, these materials do not exhibit any piezoelectric effect.

**Transversely Isotropic Materials**

Linear constitutive relations for transversely isotropic materials derived by using the procedure outlined above are given as

$$
\hat{T}^e = (\gamma_0 + \gamma_1 \hat{\theta} + \gamma_2 \mathbf{N} \cdot \mathbf{e} \mathbf{N} + \gamma_3 \mathbf{N} \cdot \mathbf{E} + 2 \gamma_4 (\text{tr} \mathbf{e})) \mathbf{1}
$$

$$
+ (\gamma_5 + \gamma_6 (\text{tr} \mathbf{e})
$$

$$
+ 2 \gamma_6 \mathbf{N} \cdot \mathbf{e} \mathbf{N} + \gamma_7 \mathbf{N} \cdot \mathbf{E} + \gamma_8 \hat{\theta}) \mathbf{N} \otimes \mathbf{N} + \gamma_9 \mathbf{e}
$$

$$
+ \gamma_{10} (\mathbf{N} \otimes \mathbf{e} \mathbf{N})_s + \gamma_{11} (\mathbf{N} \otimes \mathbf{E})_s
$$

(40)
Recall that $\mathbf{N}$ is a unit vector in the direction of transverse isotropy.

Orthotropic Materials

Linear constitutive relations for an orthotropic material with planes of symmetries perpendicular to orthonormal vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are

\[
\dot{\mathbf{T}} = (\alpha_1 + \alpha_{11} \text{tr}(\mathbf{Me}) + \alpha_{12} \mathbf{e}_1 \cdot \mathbf{e}_1 + \alpha_{13} \hat{\theta} + 2 \alpha_{16} \text{tr} \mathbf{e}) \mathbf{I} + 2 \alpha_2 \mathbf{e} + (\alpha_4 + \alpha_{11} \text{tr} \mathbf{e} + 2 \alpha_{17} \text{tr}(\mathbf{Me}) + \alpha_{16} \mathbf{e}_1 \cdot \mathbf{e}_1 + \alpha_{19} \hat{\theta}) \mathbf{M} + (\alpha_6 + \alpha_{12} \text{tr} \mathbf{e} + \alpha_{18} \text{tr}(\mathbf{Me}) + 2 \alpha_{20} (\mathbf{e}_1 \cdot \mathbf{e}_1) + \alpha_{21} \hat{\theta}) \mathbf{e}_1 \otimes \mathbf{e}_1 + 2 \alpha_3 (\mathbf{M} \otimes \mathbf{e}) + \alpha_7 \mathbf{e}_1 \otimes \mathbf{e}_1
\]

\[
\dot{\mathbf{T}}^{ne} = \gamma_9 \mathbf{e} + \gamma_{10} (\mathbf{Me})_s + \gamma_{11} (\mathbf{e}_1 \otimes \mathbf{e}_1)_s
\]

\[
\mathbf{P} = 2(\alpha_3 \mathbf{E} + \alpha_8 \mathbf{E} \mathbf{e}_1 \mathbf{e}_1)
\]

\[
\mathbf{Q} = -\beta_1 \mathbf{G} + \beta_2 \mathbf{M} \hat{\mathbf{G}} + \beta_3 (\mathbf{e}_1 \cdot \hat{\mathbf{G}}) \mathbf{e}_1
\]

\[
\eta = \eta_0 - \frac{1}{\rho_0} (\alpha_{10} + \alpha_{13} \text{tr} \mathbf{e} + \alpha_{19} \text{tr}(\mathbf{Me}) + \alpha_{21} (\mathbf{e}_1 \cdot \mathbf{e}_1) + 2 \alpha_{23} \hat{\theta})
\]

CONCLUSIONS

We derived constitutive relations for finite deformations of orthotropic, transversely isotropic, and isotropic dielectrics and deduced from them constitutive relations appropriate for infinitesimal deformations of these materials. From the constitutive relations for finite deformations, one could also derive those for small mechanical deformations, infinitesimal temperature changes, and temperature gradients but strong electric fields and also a second-order theory in which terms up to second order in displacement gradients, temperature gradients, relative temperature changes, and electric fields are kept; see, for example, Yang and Batra [14, 15].
REFERENCES