SOME BASIC FRACTURE MECHANICS CONCEPTS IN FUNCTIONALLY GRADED MATERIALS

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ABSTRACT

In this paper, the crack-tip fields in a general nonhomogeneous material are summarized. The fracture toughness and R-curve of functionally graded materials (FGMs) are studied based on the crack-bridging concept and a rule of mixtures. It is shown that the fracture toughness is significantly increased when a crack grows from the ceramic-rich region into the metal-rich region in an alumina–nickel FGM. By applying the concept of the toughening mechanism to the study of the strength behavior of FGMs, it is found that the residual strength of the alumina–nickel FGM with an edge crack on the ceramic side is quite notch insensitive. Copyright © 1996 Elsevier Science Ltd

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1. INTRODUCTION

Functionally graded materials (FGMs) for high temperature applications are special composites usually made from ceramics and metals. The ceramic in an FGM offers thermal barrier effects and protects the metal from corrosion and oxidation, and the FGM is toughened and strengthened by the metallic composition. The compositions and the volume fractions of the constituents in an FGM are varied gradually, thus giving a nonuniform micro-structure in the material with continuously graded macro-properties that are the most distinctive features of FGMs. The macro-nonhomogeneous properties of an FGM reduce thermal stresses in the material when it is subjected to thermal loading (Hasselman and Youngblood, 1978; Noda and Tsuji, 1990). Thermal residual stresses can be relaxed in a metal–ceramic layered material by inserting a functionally graded interface layer between the metal and the ceramic (Kawasaki and Watanabe, 1987; Drake et al., 1993; Giannakopoulos et al., 1995). When subjected to thermal shocks, FGM coatings suffered significantly less damage than conventional ceramic coatings (Kuroda et al., 1993; Takahashi et al., 1993).

The knowledge of the crack growth in FGMs is important in order to evaluate their integrity. Assuming an exponential spatial variation of the elastic modulus, Atkinson and List (1978), Dhaliwal and Singh (1978) and Delale and Erdogan (1983) solved crack problems for nonhomogeneous solids subjected to mechanical loads. By further assuming exponential variations of thermal properties, Jin and Noda (1993),
Noda and Jin (1993) and Erdogan and Wu (1993) computed steady thermal stress intensity factors for nonhomogeneous materials. Noda and Jin (1994) and Jin and Noda (1994a) also considered cracks subjected to transient thermal loads. Using a finite element method, Bao and Wang (1995) studied multi-cracking in an FGM coating. These works analysed the effect of material nonhomogeneities on stress intensity factors. Since the material properties or the volume fractions of the constituents of the FGMs were specifically assumed, the stress intensity factor concept can be well defined. For general nonhomogeneous materials, Schovanec and Walton (1988) and Jin and Noda (1994b) showed that the crack-tip fields are identical to those in homogeneous materials if the material properties are continuous and piece-wise continuously differentiable. Hence, the stress intensity factor concept can still be used to study the fracture behavior of FGMs as long as the crack-tip nonlinear deformations and process zones are completely included within the region dominated by the stress intensity factors. After determining the fracture toughness of FGMs, the failure analysis can be carried out. However, it is rather difficult to determine the fracture toughness of FGMs due to their complex microstructures. It is also expected that like other material properties of FGMs, the fracture toughness is also a function of the spatial position.

In this paper, we investigate some general fracture mechanics problems for FGMs. First, we summarize the crack-tip fields and give an example of the stress intensity factor in an edge-cracked strip of an FGM. Then, the fracture toughness and R-curve of the FGM are studied based on the crack-bridging concept and a rule of mixtures. Finally, the strength behavior of the FGM is examined.

2. CRACK-TIP STRESS FIELDS

Functionally graded materials are multiphase materials with spatially varying properties tailored to satisfy specific requirements encountered in engineering applications. From continuum mechanics viewpoint, FGMs are nonhomogeneous materials. For a nonhomogeneous material undergoing plane strain elastic deformations, the Airy stress function $F$ satisfies

$$\nabla^2 \left( \frac{1}{E} \nabla^2 F \right) - \frac{\partial^2}{\partial y^2} \left( \frac{1}{2\mu} \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2}{\partial x^2} \left( \frac{1}{2\mu} \frac{\partial^2 F}{\partial y^2} + 2 \frac{\partial^2}{\partial x \partial y} \left( \frac{1}{2\mu} \frac{\partial^2 F}{\partial x \partial y} \right) \right) = 0. \quad (1)$$

Equation (1) is written in rectangular Cartesian coordinates. Here $E$ is Young’s modulus, $\nu$ Poisson’s ratio, $\mu$ the shear modulus and $\nabla^2$ the Laplacian operator. Jin and Noda (1994b) showed that when $E$ and $\nu$ are continuous and piece-wise continuously differentiable, the crack-tip elastic fields are identical to those in homogenous materials, i.e. at the crack tip,

$$\sigma_{sp}(r, \theta) = \frac{1}{\sqrt{2\pi r}} \left\{ K_1 \sigma_{sp}^{(1)}(\theta) + K_{II} \sigma_{sp}^{(2)}(\theta) \right\}, \quad r \to 0. \quad (2)$$

where $\sigma_{sp}$ are stresses, $(r, \theta)$ are the polar coordinates with origin at the crack tip, $K_1$ and $K_{II}$ are mode I and mode II stress intensity factors, respectively, and $\sigma_{sp}^{(1)}(\theta)$ and $\sigma_{sp}^{(2)}(\theta)$ are standard angular distribution functions, which can be found in any fracture.
mechanics book, for example, Liebowitz (1968). For plane strain deformations, the crack-tip energy release rate $G$ is related to the stress intensity factors by

$$G = \frac{1 - v_{\text{tip}}^2}{E_{\text{tip}}} (K_1^2 + K_2^2),$$

where $E_{\text{tip}}$ and $v_{\text{tip}}$ are Young’s modulus and Poisson’s ratio for the material at the crack tip. Thus, we can still use stress intensity factor and fracture toughness concepts to study the fracture behavior of FGMs. Schovanec and Walton (1988) also showed that the mode III elastic crack-tip stress singularity is identical to that of a homogeneous material. We note that the dominant parts of the crack-tip stresses in a material with shear modulus $\mu = \mu_0 \exp(\beta x + \gamma y)$ given by Konda and Erdogan (1994) and Erdogan (1995) are also given by (2).

The asymptotic solution (2) is only valid at points very close to the crack tip as compared to the crack length or any other characteristic length of the cracked body. Whereas gradients of the moduli of elasticity do not influence the square-root singularity (2), they may affect the size of the region in which (2) holds. A rough estimate of their effect may be made on the basis of (1) and (2) as we have not considered higher order terms in the asymptotic stress field. The first term in (1) equals

$$\frac{1 - v^2}{E} \nabla^2 \nabla^2 F + 2 \left[ \frac{1 - v^2}{E} \frac{\partial^2 F}{\partial x^2} + \frac{1 - v^2}{E} \frac{\partial^2 F}{\partial y^2} \right] + \nabla^2 \left( \frac{1 - v^2}{E} \right) \nabla^2 F. \quad (4)$$

For a mode I crack, at a radial distance $r$ from the crack tip,

$$\nabla^2 F \sim \frac{2K_1}{\sqrt{2\pi r}}, \quad \frac{\partial^2 F}{\partial x \partial x} \sim \frac{K_1}{\sqrt{2\pi r}} \quad (5)$$

to within an angular multiplier of order unity. Neglecting gradients of Poisson’s ratio, the crack-tip solution (2) will be valid at points for which

$$1 \frac{\partial E}{E \partial x} \ll \frac{1}{r}, \quad 1 \frac{\partial^2 E}{E \partial x \partial x} \ll \frac{1}{r^2}. \quad (6)$$

Let $r_p$ be the size of the crack-tip nonlinear deformations and damage zone. Thus for a $K$-dominant zone to exist near the crack tip,

$$1 \frac{\partial E}{E \partial x} \ll \frac{1}{r_p}, \quad 1 \frac{\partial^2 E}{E \partial x \partial x} \ll \frac{1}{r_p^2}. \quad (7)$$

The determination of $r_p$ requires a detailed investigation of the crack-tip nonlinear deformations and damage in FGMs. The results for metals are not necessarily applicable to FGMs since the deformation and damage mechanism in FGMs is different.

3. THE STRESS INTENSITY FACTOR FOR AN EDGE CRACK IN A STRIP OF AN FGM

Consider plane strain deformations of an FGM strip containing an edge crack and subjected to pure bending moment $M$ at infinity as shown in Fig. 1. The surface
cracking is a common failure mode in FGMs (Kawasaki and Watanabe, 1993 and Takahashi et al., 1993).

Assume that

\[
\mu = \mu_0[1 + \beta(x/b)], \tag{8a}
\]

\[
\nu = 1 - (1 - \nu_0)e^{i(x/a_0)}[1 + \beta(x/b)], \tag{8b}
\]

\[
E = E_0 \frac{2[1 + \beta(x/b)] - (1 - \nu_0)e^{i(x/a_0)}}{(1 + \nu_0)[1 + \beta(x/b)]^2}. \tag{8c}
\]

where \(E_0\), \(\nu_0\) and \(\mu_0\) are Young's modulus, Possion's ratio and the shear modulus at \(x = 0\), \(b\) is the width of the specimen and constants \(\beta\) and \(\gamma\) are given by

\[
\beta = \frac{\mu_0}{\mu_i} - 1, \tag{9a}
\]

\[
\gamma = \ln(1 + \beta) + \frac{1 - \nu_1}{1 - \nu_0}. \tag{9b}
\]

in which \(\nu_1\) and \(\mu_i\) are Poisson's ratio and the shear modulus at \(x = b\).

The volume fractions of the constituents in a metal–ceramic FGM with its properties given by (8) and (9) may be obtained with an appropriate micro-mechanical model. Conventional composite models may be used when the volume fraction of one constituent in the FGM is much smaller than that of the other one. However, their validity cannot be assured over the entire range of constituent volume fractions because they were developed on the assumption that the constituent distributions and micro-structure are uniform in composites. The main feature of FGMs is the nonuniform micro-structure with the continuous change in constituent volume fractions. In fact, micro-mechanical models appropriate to FGMs are not presently available (Zuiker, 1995). According to the criteria given by Zuiker (1995), the three-
phase model (TPM) or the so-called generalized self-consistent models (Smith, 1975; Christensen and Lo, 1979) is better than other models. The TPM was also used by Bao and Wang (1995) to determine approximately the effective properties of an FGM coating. We assume that the FGM being studied is obtained by dispersing the metallic particles within a ceramic matrix. Since the volume fraction can generally be determined with only one assumed property, other properties predicted by the TPM will, in general, be different from the assumed ones. Consider the alumina (Al$_2$O$_3$)-nickel alloy (Ni) FGM which is typical in FGM studies (Rabin and Heaps, 1993; Drake et al., 1993; Giannakopoulos et al., 1995) with the following constituent properties

$$E_1 = 200 \text{ GPa}, \quad v_1 = 1/3, \quad \mu_1 = 75 \text{ GPa}$$

for the metal phase, and

$$E_0 = 360 \text{ GPa}, \quad v_0 = 0.2, \quad \mu_0 = 150 \text{ GPa}$$

for the ceramic phase. Thus constants $\beta$ and $\gamma$ in (8) and (9) equal 1 and 0.5158, respectively.

It is assumed that the volume fraction $V_m$ of the metallic phase equals 0 at $x = 0$ and 1 at $x = b$. Figure 2(a) shows the spatial variation of the shear modulus of the FGM and of the volume fraction of the metal phase. Since the volume fraction is determined with the assumed shear modulus, the shear modulus predicted by the TPM is identical to the assumed one. Spatial variations of Young’s modulus and Poisson’s ratio depicted in Fig. 2(b) and (c), respectively, show that the assumed Young’s modulus (8c) and Poisson’s ratio (8b) agree reasonably well with those computed from the TPM. It is seen from Fig. 2(c) that the assumed Poisson’s ratio satisfies the constraint $0 \leq \nu \leq 0.5$. Of course, the TPM and other micro-mechanical models developed for macro-homogeneous composites are only approximately valid for FGMs.

The integral equation for the crack problem can be written as

$$\int_s^b \left[ \frac{1}{s-r} + K(r,s) \right] \phi(s)e^{-(r-b)(1+\gamma)} ds = -\frac{2\pi(1-v_0^2)}{E_0} \frac{\gamma^2 e^{-\gamma \alpha}}{6A_0} \cdot \left[ \frac{\gamma^2 \alpha}{2b(1+\gamma)} A_{11} - A_{12} \right] \frac{6M}{b^2}$$

(10a)

in which $\phi(r)$ is the dislocation density along the crack face defined by

$$\phi(x) = \frac{\partial \gamma(x,0)}{\partial x}.$$

(10b)

$\gamma(x,0)$ is the displacement in $y$-direction at the crack surface, $K(r,s)$ is a Fredholm type kernel (Jin and Batra, 1996), $r = (2x/a - 1)$, $s = (2x'/a - 1)$, and constants $A_0$ and $A_i$ ($i, j = 1, 2$) are given by

$$A_{11} = 1 - e^{-\gamma},$$

$$A_{12} = A_{21} = 1 - (1+\gamma)e^{-\gamma},$$

$$A_{22} = 2 - [1 + (1+\gamma)]e^{-\gamma}.$$
Fig. 2. (a) The variation with respect to $x$ of (a) the shear modulus of the FGM and the corresponding volume fraction of the metal in the FGM by the TPM; (b) Young’s modulus predicted from the TPM and the present assumption (8c); (c) Poisson’s ratio predicted from the TPM and the present assumption (8b).
\[ A_0 = A_{11}A_{22} - A_{12}A_{21} = 1 - (2 + \gamma^2)e^{-\gamma} + e^{-2\gamma} \]  

(11)

According to the singular integral equation method (Muskhelishvilli, 1965; Gupta and Erdogan, 1974), (10) has a solution of form

\[ \phi(r) = e^{\alpha(b(1 + r)^{-1}) \cdot \frac{\Psi(r)}{\sqrt{1 - r}}}, \]  

(12)

where \( \Psi(r) \) is a continuous and bounded function in the interval \([-1, 1]\). If \( \Psi(r) \) is normalized by \( \sigma_b(1 - \nu^2)/E_b \), then the normalized stress intensity factor \( K^* \) at the crack tip is given by

\[ K^* = \frac{K_I}{\sigma_b \sqrt{\pi a}} = -\frac{1}{2} \Psi(1), \]  

(13)

where

\[ \sigma_b = 6M/b^2. \]  

(14)

Figure 3 shows the normalized stress intensity factor (SIF), \( K^* \), versus the normalized crack length \( a/b \) for both the FGM and homogeneous materials. It is seen that the SIF for the FGM varies with \( a/b \) in the same way as for homogeneous materials, but the SIF of the FGM is higher than that of homogeneous materials. This is opposite to the results for thermal loads where the thermal SIF is reduced (Jin and Batra, 1996). However, the reduction in thermal SIF is largely due to the thermal conductivity gradient. Though the SIF in the FGM is increased, it will be seen that this deleterious effect is completely offset by the high fracture toughness of the FGM and as a result, the residual strength of the cracked FGM is much higher than that of the pure ceramic; we discuss this below.

The edge crack problem considered here has also been studied by Erdogan and Wu (1993) on the assumptions that

![Fig. 3. Normalized stress intensity factors of the edge cracked strips of the FGM and homogeneous materials.](image-url)
\[ \mu = \mu_0 e^{\alpha x}, \quad \nu = \nu_0 \] 

and their variation of the SIF with \( a/b \) is similar to that shown in Fig. 3.

4. FRACTURE TOUGHNESS AND R-CURVE

It has been experimentally observed that surface cracking in the material gradient direction at the ceramic side is the most common failure mode of a metal–ceramic FGM when it is subjected to a thermal shock (Kawasaki and Watanabe, 1993; Shinohara et al., 1993; Takahashi et al., 1993). Another major failure mode is cracking perpendicular to the material gradient direction which corresponds to delamination as real FGMs usually are multi-layered materials (Kawasaki and Watanabe, 1993; Rabin and Heaps, 1993; Tanihara et al., 1993) and the crack grows along the interface. In both cases, the crack may deflect from the original direction and then grow in a different direction. In our study of R-curve behavior of FGMs, we are only concerned with the mode I crack growing in the direction of the material gradation. Also, it is assumed that the crack will grow from the ceramic-rich region into the metal-rich region as cracks are more likely generated first in the ceramic-rich region.

4.1. Fracture toughness based on a rule of mixtures

A simple way to determine the fracture toughness of a metal–ceramic FGM is to use a rule of mixtures. In general, the toughness may be overestimated by a rule of mixtures as pointed out by Krstic (1983) since the fracture toughness of a metal in the bulk form is much larger than that for the corresponding particulates dispersed in a brittle matrix.

If a crack grows through both the ceramic and metal phases without any debonding occurring and the metal fractures in a brittle manner, then a rule of mixtures holds for the surface energy or the critical energy release rate. Assuming that the crack growth and material gradation are in the \( x \)-direction, the critical energy release rate for the FGM may be expressed as

\[ G_{IC}(x) = V_m(x)G^\text{metal}_{IC} + (1 - V_m(x))G^\text{ceram}_{IC}, \]  

(16)

where \( G_{IC}(x) \) is the critical energy release rate of the FGM, \( G^\text{ceram}_{IC} \) and \( G^\text{metal}_{IC} \) are the critical energy release rates of the ceramic and metal, respectively, and \( V_m(x) \) is the volume fraction of the metal. The value of \( V_m \) here equals the area fraction intercepted by the crack, which is supported by experiments of Sigl et al., (1988). Noting that

\[ G_{IC}(x) = \frac{1 - v^2(x)}{E(x)} K_{IC}^2, \]  

(17a)

\[ G^\text{metal}_{IC} = \frac{1 - v_1^2}{E_1} (K^\text{metal}_{IC})^2, \]  

(17b)

\[ G^\text{ceram}_{IC} = \frac{1 - v_0^2}{E_0} (K^\text{ceram}_{IC})^2, \]  

(17c)
where $K_{IC}(x)$, $K_{IC}^{\text{ceram}}$, and $K_{IC}^{\text{metal}}$ are the fracture toughnesses of the FGM, ceramic and the metal, respectively, $E_0$ and $v_0$ are Young’s modulus and Poisson’s ratio of the ceramic, and $E_1$ and $v_1$ are Young’s modulus and Poisson’s ratio of the metal, we obtain the following for the fracture toughness of the FGM.

\[
\frac{K_{IC}(x)}{K_{IC}^{\text{ceram}}} = \left\{ E(x) \left[ V_m(x) \frac{1-v_1^2}{E_1} \left( \frac{K_{IC}^{\text{metal}}}{K_{IC}^{\text{ceram}}} \right)^2 + (1-V_m(x)) \frac{1-v_0^2}{E_0} \right] \right\}^{1.2}. \tag{18}
\]

Equation (18) implies that the fracture toughness of the FGM varies with $x$. When a crack grows from the ceramic-rich region into the metal-rich region, it is expected that the toughness of the FGM will increase significantly since the fracture toughness of the metal is usually much higher than that of the ceramic. However, it should be noted that this increase in toughness predicted by (18) is just due to the existence of the gradation of the fracture toughness. Krstic (1983) has pointed out that the fracture toughness of a metal in the bulk form is much larger than that of the corresponding particulates dispersed in a brittle matrix. Hence, the metal toughness $K_{IC}^{\text{metal}}$ appearing in (18) for the particulate form may be significantly lower than that for the bulk form. As a result, the fracture toughness of the FGM may be overestimated by (18) when the fracture toughness of the bulk metal is used.

4.2. Fracture toughness based on the crack-bridging concept

Crack bridging is believed to be a major toughening mechanism in ductile particulate reinforced brittle matrix composites (Krstic, 1983; Evans and McMeeking, 1986; Budiansky et al., 1988; Sigl et al., 1988; Bao and Hui, 1990). When a metal–ceramic FGM is fabricated with the metal having relatively large grains and the ceramic having smaller grains (Rabin and Heaps, 1993) the FGM may be regarded as a metal particulate reinforced ceramic composite with graded compositions. In this case, the crack-bridging concept may be used to study the toughness behavior or $R$-curve of the FGM. However, the crack-bridging concept may not be always appropriate to FGMs as the microstructure of an FGM is different from that of a particulate composite. We assume here that the grain size of the metal in an FGM is much larger than that of the ceramic so that the crack-bridging concept can be applied.

Consider metal–ceramic FGM strip of width $b$ with an initial edge crack of length $a_0$ as shown in Fig. 1. Once the crack initiates, it will grow in the ceramic with plastically stretched metal grains behind the crack tip bridging the crack faces. It is assumed that the metal elsewhere is not deformed plastically. In order to calculate the fracture toughness or $R$-curve due to the ductile particulate bridging, one needs a bridging law to relate the bridging stress $\sigma$ to the crack opening displacement $\delta$ of the bridging zone, $\sigma = \sigma(\delta)$. Previous studies (Sigl et al., 1988; Mataga, 1989; Bao and Hui, 1990) showed that for ductile particulate reinforced brittle matrix composites, the bridging exhibited softening behavior, i.e. the bridging stress decreases with an increase in the crack opening. The bridging law then can be modeled by

\[
\frac{\sigma}{\sigma_0} = \left( 1 - \frac{\delta}{\delta_0} \right)^n, \tag{19}
\]
where $\sigma_0$ is the maximum bridging stress, $\delta_0$ is the maximum crack opening displacement of the bridging zone at which the bridging stress drops to zero and $n$ is the softening index. Mai and Lawn (1987) used (19) to describe the grain bridging behavior in polycrystalline ceramics. We here consider a linear softening case and set the softening index $n$ equal to 1. Suo et al. (1993) argued that the residual strength is insensitive to the form of the bridging law. Jin and Mai (1995a) showed that the thermal shock residual strength of ceramics is insensitive to the softening index $n$ provided the maximum bridging stress $\sigma_0$ and the bridging energy, $G_b$, defined by

$$G_b = \int_0^{\delta_0} \sigma(\delta) \, d\delta \quad (20)$$

remain unchanged. In their study of the strength of ductile particulate composites, Bao and Zok (1993) replaced the complex bridging law derived by Mataga (1989) by a linear softening one.

Following a procedure similar to that described in Section 3 and considering the crack face bridging described by (19) with $n = 1$, we obtain the following integral equation for the dislocation density $\phi$ along the crack face

$$\int_{-1}^{1} \left[\frac{1}{s-r} + K(r,s)\right] \phi(s) e^{-(\omega_0)(1+s)/2b} \, ds - H(r-r_0) V_m(r) \frac{a}{a_0} \varepsilon \int_{r}^{1} \phi(s) \, ds$$

$$= -\frac{2\pi(1-v_0^2)}{E_0} \left[\frac{e^{-1/2}(\omega_0)(1+r)}{6A_0} \left(\gamma^2 A_1 + \frac{1}{2} a (1+r) - \gamma^2 A_2 \right) \sigma_b \right.$$\n
$$- H(r-r_0) V_m(r) \sigma_0 \left.\right] \quad (21)$$

Here $H(\cdot)$ is the Heaviside step function, $r_0 = 2a_0/a - 1$. $a_0$ is the initial crack length, $a = a_0 + \Delta a$ is the current crack length, $\Delta a$ is the bridging length, and $\varepsilon$ is a nondimensional parameter given by

$$\varepsilon = \frac{2\pi a_0 (1-v_0^2)}{E_0 \delta_0 / \sigma_0} \quad (22)$$

The crack opening $\delta$ is related to $\phi$ by

$$\delta(r) = -a \int_{r}^{1} \phi(s) \, ds \quad (23)$$

The solution of (21) also has the form of (12) (Jin and Mai, 1995b) and can be written as

$$\phi(r) = e^{(\omega_0)(1+r)/2b} \frac{\psi(r)}{\sqrt{1-r}} = \frac{e^{(\omega_0)(1+r)/2b}}{\sqrt{1-r}} \left[\frac{1-v_0^2}{E_0} \sigma_b \psi_1(r) + \frac{1-v_0^2}{E_0} \sigma_0 \psi_2(r) \right] \quad (24)$$

where $\psi_1(r)$ and $\psi_2(r)$ are due to $\sigma_b$ and $\sigma_0$, respectively. The stress intensity factor $K_{\text{tip}}$ at the crack tip $x = a$ can be evaluated from
\[ K_{\text{tip}} = -\frac{1}{2} \sqrt{\pi a} [\sigma_b \psi_1(1) + \sigma_a \psi_2(1)]. \] (25)

When \( K_{\text{tip}} \) reaches a critical value, the crack will grow in the ceramic but will be bridged by the metal particulates. The crack growth is assumed to be maintained by

\[ G_{\text{tip}} = (1 - V_m(a)) G_{\text{ceram}} = (1 - V_m(a)) \frac{1 - \nu_0^2}{E_0} (K_{\text{IC,ceram}})^2, \] (26)

where \( G_{\text{tip}} \) is the crack-tip energy release rate which is related to \( K_{\text{tip}} \) by

\[ G_{\text{tip}} = \frac{1 - \nu^2(a)}{E(a)} K_{\text{tip}}^2. \] (27)

Hence, we have the crack growth condition

\[ K_{\text{tip}} = K_c = \left\{ (1 - V_m(a)) \frac{1 - \nu_0^2}{1 - \nu^2(a)} \frac{E(a)}{E_0} \right\}^{1/2} K_{\text{IC,ceram}}. \] (28)

The bending stress \( \sigma_b \) corresponding to \( K_{\text{tip}} = K_c \) can be obtained from (25) as

\[ \frac{\sigma_b(a)}{\sigma_0} = -\frac{1}{2} \psi_1(1) \left\{ \frac{K_{\text{IC,ceram}}}{\sigma_0 \sqrt{\pi a}} \left[ (1 - V_m(a)) \frac{1 - \nu_0^2}{1 - \nu^2(a)} \frac{E(a)}{E_0} \right]^{1/2} + \frac{\psi_2(1)}{2} \right\}. \] (29)

The \( R \)-curve can then be evaluated by

\[ \frac{K_R(a)}{K_{\text{IC,ceram}}} = \frac{\sigma_0 \sqrt{\pi a}}{K_{\text{IC,ceram}}} \left( \frac{\sigma_b}{\sigma_0} \right) \left( -\frac{1}{2} \right) \phi^0(r) \bigg|_{r=1}, \] (30)

where \( \phi^0(r) = e^{-(a/b)(1 + r)^{2/3}}(1 - r)^{1/2} \phi^0(r) \) and \( \phi^0(r) \) is the solution of (21) without considering bridging, which is the same as (10).

Figure 4 shows the fracture toughness and \( R \)-curve based on (18) and (30) for the

![Fracture toughness of the FGM versus the normalized crack length.](image-url)
Al$_2$O$_3$-Ni FGM considered in Section 3 with $K_{IC}^{cera}$ = 2 MPa$\sqrt{m}$ and $K_{IC}^{metal}$ = 100 MPa$\sqrt{m}$. The yield stress and the strain hardening index of the nickel alloy equal 800 MPa and 0.11, respectively (Everhart, 1971). Following Bao and Zok (1993) the maximum bridging stress $\sigma_0$ is taken as the maximum value in the model of Mataga (1989) and the maximum crack opening $\delta_0$ of the bridging zone is determined by equating the bridging energy of the model of Mataga (1989) to $G_0 = \sigma_0 \delta_0/2$, the bridging energy of the linear softening model (19) with $n = 1$. The values $\sigma_0$ and $\delta_0$ are thus evaluated as 1290 MPa and 33 $\mu$m, respectively, when the average grain size of the metal is 20 $\mu$m. In order to study the effect of the initial flaw size on the $R$-curve of the FGM, three initial crack lengths, $a_0/b$ = 0.01, 0.1 and 0.2 are considered with specimen width $b$ = 10 mm. Figure 4 shows that the FGM exhibits significant $R$-curve behavior as the crack grows from the ceramic-rich region into the metal-rich region. It is clear that the predicted toughness based on the rule of mixtures (18) is significantly higher than that based on the crack-bridging concept. As pointed out earlier, the rule of mixtures overestimates the fracture toughness. It is evident from Fig. 4 that based on the crack-bridging concept, the initial crack size influences strongly the slope of the $R$-curve especially during the initial stage of the crack growth. But when the crack has grown so that the bridging zone is fully developed, i.e. the crack opening at the tail of the bridging zone reaches $\delta_0$, the toughness reaches nearly the same value for all initial crack sizes considered. Because of the significant effect of the initial crack size on the $R$-curve, it is not a material property.

For large values of the metal volume fraction, the crack bridging may not be the only major mechanism influencing the fracture toughness and the residual strength (discussed below) of the FGM. In that case, extensive plastic deformation of the metal, microcracking and other forms of damage in the ceramic, and the debonding between the ceramic and the metal may occur in a diffusive region around the crack tip. Further work is needed to understand the constitutive behavior in that region.

5. RESIDUAL STRENGTH

Based on the toughening mechanism from which the fracture toughness and $R$-curve are derived, we can evaluate the residual strength of the FGM with an edge crack (cf. Fig. 1). Here we do not use the $R$-curve to calculate the residual strength as it is not a material property. When a rule of mixtures is used, the residual strength, $\sigma_R$, can be obtained by equating the stress intensity factor $K_i$ in (13) to the fracture toughness (18) and taking the maximum of the bending stress during the subsequent crack growth.

$$\sigma_R(a_0) = \max_{a \geq \delta_0} \left\{ \frac{1}{2} \psi(1) \frac{K_{IC}(a)}{\pi a} \right\},$$  \hspace{1cm} (31)

where $K_{IC}(a)$ is given by (18) and $\psi(1)$ is the solution of (10) and (12).

Based on the crack-bridging concept, the residual strength is obtained from (29) as
\[
\sigma_R(a_0) = \frac{\sigma_0}{\max_{a > a_0} \left\{ \frac{\sigma_0(a)}{\sigma_0} \right\}}
\]

with \( \sigma_0(a) \) given by (29).

Figure 5 shows that the residual strength of the Al\(_2\)O\(_3\)--Ni FGM based on both the crack-bridging concept and the rule of mixtures is insensitive to the normalized crack length \( a_0/b \). The residual strength computed from the crack-bridging concept remains at a nearly constant value of 400 MPa until \( a_0/b = 0.1 \) and then decreases to 216 MPa at \( a_0/b = 0.5 \). However, based on the rule of mixtures, it equals about 700 MPa until \( a_0/b = 0.1 \) and then drops to 420 MPa for \( a_0/b = 0.5 \). The residual strength based on the rule of mixtures is much higher than that based on the crack-bridging concept. We believe that the rule of mixtures overestimates the residual strength and that based on the crack-bridging concept is more reasonable for the FGM considered. Also shown in Fig. 5 is the residual strength of pure Al\(_2\)O\(_3\) which is very sensitive to the initial crack length as is well-known and is much lower than that of the FGM.

6. CONCLUSIONS

Some general fracture mechanics problems for functionally graded materials are studied. The crack-tip fields in FGMs are identical to those in homogeneous materials. The fracture toughness can be significantly increased when a crack grows from the ceramic-rich region into the metal-rich region in an Al\(_2\)O\(_3\)--Ni FGM based on the crack-bridging concept as well as a rule of mixtures. By applying these concepts to study the strength behavior of FGMs the residual strength of the Al\(_2\)O\(_3\)--Ni FGM with an edge crack at the side of the ceramic is evaluated and is found to be notch insensitive. Both the fracture toughness and the residual strength of the FGM may be overestimated by the rule of mixtures and results based on the crack-bridging concept are believed to be more realistic.
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