Generalized Plane Solution for Monoclinic 
Piezoelectric Laminates

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An efficient method has been developed for analyzing the three-dimensional electromechanical deformations of a monoclinic piezoelectric laminate subjected to surface tractions and electric displacements on the top and bottom surfaces. The procedure combines the transfer matrix method with the asymptotic expansion technique. The electromechanical coupled formulation is reduced to a hierarchy of problems referred to as a reference plane. A closed-form solution is obtained for the generalized plane deformations of a piezoelectric laminate with the top and bottom surfaces subjected to uniform electric displacements and mechanical normal tractions and edges at the midplane rigidly clamped and grounded. The closed-form solution is obtained because the asymptotic expansion terminates after a few terms. For some loadings the in-plane electric fields can be much larger than the out-of-plane electric fields.

I. Introduction

A SMART structural system is a composite structure with surface-mounted or embedded active materials. Beams, plates, and shells are the structural elements most frequently used in aeronautics and aerospace applications. The integrated active structures contain piezoelectric patches and layers that act as sensors and actuators. References 1–4 provide more details of the immense technological potentials and implications of the smart materials and structures.

A piezoelectrically induced strain as a source of actuation may be interpreted as an eigenstrain.6,9 entering the constitutive relations. These approaches estimate eigenstrain from the applied voltage and provide no information on the errors involved. Moreover, in certain circumstances11–13 the in-plane electric field components are more significant than the transverse component, thereby making their omission unjustifiable.

Transfer matrix approaches14–19 have been developed to study the electromechanical coupling characteristics of laminated piezoelectric plates. However, the basic equations must be reducible to a system of ordinary differential equations in which the thickness coordinate is the only spatial variable. This restricts applications of the method to a small class of problems. An asymptotic expansion method for three-dimensional elastic analysis20–22 has been successfully developed. An important result is that solutions of the three-dimensional elasticity equations can be generated from solutions of the classical two-dimensional plate equations. In piezoelectricity an asymptotic theory of leading-order approximation for thin single-layer homogeneous piezoelectric plates23,24 has also been proposed. By extending the elasticity work,20 a three-dimensional solution of electroelastic elliptic symmetric laminates has been presented. However, all of these electroelasticity approaches appear to be applicable to plates symmetric about the midplane.

Here, the three-dimensional asymptotic approach is further developed for monoclinic piezoelectric laminates, which are not necessarily symmetric about the midplane. General formulations are derived to any successive approximations and used for a generalized plane problem. Neglecting the effects of the boundary layer near the clamped edges, a closed-form solution is obtained at interior points of a piezoelectric laminate subjected to uniform normal tractions and electric displacements on the top and bottom surfaces. This solution essentially generalizes the elasticity solution.21

II. Formulation of the Problem

We use rectangular Cartesian coordinates xi, (i = 1, 2, 3) to describe the infinitesimal electromechanical deformations of a plate of uniform thickness h and take the midplane of the plate to coincide with the plane x3 = 0. The plate is made of a monoclinic piezoelectric material. In the absence of body forces and electric charge density, equations governing the quasistatic electromechanical deformations of the body are25–27

\[ \tau_{ij} = 0, \quad D_{ij} = 0 \]  

(1)

where \( \tau_{ij} \) is the stress tensor and \( D_{ij} \) the electric displacement. Throughout this paper a comma followed by an index i denotes partial differentiation with respect to \( x_i \), a repeated index implies summation over the range of the index, and Latin indices range from 1 to 3 and Greek indices from 1 to 2. The dependence of functions and operators on \( x_i \) is not explicitly shown unless necessary. The infinitesimal strain tensor \( S_{ij} \) and the electric field \( E_j \) are related to the mechanical displacements \( u_i \) and the electric potential \( \varphi \) through the gradient relations

\[ S_{ij} = \frac{1}{2} (u_{ij,i} + u_{ij,i}), \quad E_k = -\varphi x_k \]  

(2)

The constitutive relations for a monoclinic piezoelectric material can be written as

\[ \tau_{ij} = c_{ijkl} S_{kl} + e_{ijkl} E_j \]  

\[ \tau_{ij} = 2 c_{ijkl} S_{kl} - e_{ijkl} E_j \]  

(3)

\[ \tau_{ij} = c_{ijkl} S_{kl} + e_{ijkl} E_j \]  

Here, we have assumed that the material properties are symmetric with respect to the \( x_1 \times x_2 \) plane, i.e., the diad axis is the \( x_3 \) direction. Accordingly, the numbers of nonzero independent elastic moduli \( c_{ijkl} \), piezoelectric moduli \( e_{ijkl} \), and dielectric moduli \( e_{ik} \) are respectively 13, 8, and 4 for the monoclinic material. These material moduli exhibit the following symmetries:

\[ c_{ijkl} = c_{jikl} = c_{klij}, \quad e_{ijkl} = e_{jkli}, \quad e_{ik} = e_{ki} \]  

(4)
For a laminated plate with laminate made of different homogeneous monoclinic piezoelectric materials, the material moduli are piecewise constant functions of \( x_3 \).

By eliminating the possibly discontinuous in-plane stresses \( \tau_{ij} \) and in-plane electric displacements \( D_i \) from Eqs. (1) and (3), and expressing \( S_{3ik} \) in Eq. (3) in terms of \( \tau_{ij} \) and \( \varphi \), a state-space equation for the piezoelectric plate is formulated as

\[
\partial_3 \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix}
\]

(5)

where \( \partial_3 \equiv \partial/\partial x_3 \) and the state-space variables are chosen as

\[
F = \begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \\ \varphi \end{bmatrix}, \quad G = \begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \\ \varphi \end{bmatrix}
\]

(6)

The \( 4 \times 4 \) matrices \( A \) and \( B \) in the state-space equation (5) are differential operator matrices:

\[
A = \begin{bmatrix} I & J_{\alpha} \end{bmatrix}, \quad B = \begin{bmatrix} -L_{\alpha} & -M_{\alpha} \\ 0 & N \end{bmatrix}
\]

(7)

where both \( A \) and \( B \) have been partitioned into four \( 2 \times 2 \) operator submatrices and

\[
I \equiv (I^{\alpha\alpha}) = \begin{bmatrix} I_{11}^{\alpha} & I_{12}^{\alpha} \\ I_{21}^{\alpha} & I_{22}^{\alpha} \end{bmatrix}, \quad J_{\alpha} \equiv (J_{\alpha}^{\alpha\alpha}) = \begin{bmatrix} J_{11}^{\alpha} & J_{12}^{\alpha} \\ J_{21}^{\alpha} & J_{22}^{\alpha} \end{bmatrix}
\]

(8)

For a laminated plate \( A \) and \( B \) are piecewise constant functions of \( x_3 \); the submatrices are only related to the material moduli, and their elements can be scalars, vectors, and tensors. We use superscripts to denote the location, i.e., row and column, of matrix elements in Eq. (8), whereas the Greek subscripts are for the usual tensor notation. These elements are expressed in terms of the material moduli as

\[
J_{11}^{\alpha} = J_{22}^{\alpha}, \quad J_{12}^{\alpha} = J_{21}^{\alpha} = 0, \quad K_{\alpha\alpha}^{12} = K_{\alpha\alpha}^{21} = 0, \quad L_{\alpha\alpha}^{12} = L_{\alpha\alpha}^{21} = 0
\]

(9)

where \( \delta_{\alpha\beta} \) is the Kronecker delta and

\[
I^{\alpha} = \begin{bmatrix} C_{111}^{\alpha} & C_{122}^{\alpha} & C_{123}^{\alpha} \\ C_{122}^{\alpha} & C_{112}^{\alpha} & C_{223}^{\alpha} \end{bmatrix}, \quad N^{\alpha} = \begin{bmatrix} C_{333}^{\alpha} & C_{331}^{\alpha} & C_{331}^{\alpha} \\ C_{331}^{\alpha} & C_{333}^{\alpha} & -C_{331}^{\alpha} \end{bmatrix}
\]

(10)

The in-plane stresses and in-plane electric displacement, which may be discontinuous in \( x_3 \), are then given by

\[
\tau_{ij} = L_{\alpha\alpha}^{ij} e_{ij\alpha} + M_{\alpha} C_{333} + M_{\alpha}^{ij2} D_3
\]

(11)

\[
D_i = J_{\alpha}^{ij} \tau_{ij} - K_{\alpha\alpha}^{ij2} \phi
\]

(12)

The state-space equation (5) for piezoelectricity is structurally the same as that for pure elasticity except for the contribution from the two additional variables because of the electric field. The state-space equation for an elastic plate can be recovered from Eq. (5) by setting \( e_{ij\alpha} \equiv 0 \).

We introduce a dimensionless parameter \( \varepsilon = h/2a \), where \( a \) is a typical in-plane length of the plate and set \( z = x_3/\varepsilon \). Thus \( z \) varies from \(-a\) to \( a \) as \( x_3 \) goes from \(-h/2\) to \( h/2 \). The state-space equation (5) can now be written as

\[
\partial_z \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix}
\]

(13)

and has the formal solution \( F = e \begin{bmatrix} A \\ B \end{bmatrix} G \)

(14)

where \( z^* \) takes a specific value of \( z \) and hence the initial functions \( F(z^*) \) and \( G(z^*) \) are functions of \( x_3 \) only. Their components are the unknowns to be determined from specified surface and edge conditions. The transfer matrix \( P \) is given by

\[
P = \sum_{n = 0}^{\infty} e^{n\varepsilon z^*} \begin{bmatrix} a^{(2n)} & a^{(2n + 1)} \\ b^{(2n)} & b^{(2n + 1)} \end{bmatrix}
\]

(15)

where

\[
a^{(n)} = b^{(n)} = 0, \quad n \geq 0
\]

(16)

is a \( 4 \times 4 \) identity matrix. Similar to \( A \) and \( B \), \( a^{(n)} \) and \( b^{(n)} \) are the operator matrices that include differential operators only with respect to \( x_3 \), whereas their variation with \( z \) appears only in the material moduli.

We expand the state-space variables \( F \) and \( G \) in terms of the small parameter \( \varepsilon \)

\[
F = \sum_{n = 0}^{\infty} e^{n\varepsilon z^*} \begin{bmatrix} q^{(n)} \\ g^{(n)} \end{bmatrix}
\]

(17)

Substitution from Eqs. (14) and (17) into Eq. (13) yields

\[
f^{(n)} = \sum_{k = 0}^{\infty} a^{(2k)} f^{(n - k)}(z^*) + a^{(2k + 1)} g^{(n - k)}(z^*)
\]

(18)

Equations (18) furnish general expressions for the \( n \)th-order expansion coefficients of the field functions in terms of the expansion coefficients of the initial functions from the zeroth order to the \( n \)th order. Hence the three-dimensional problem reduces to a series of two-dimensional problems defined on the reference plane. As for most plate theories, we designate the reference plane \( z = z^* \) to be the midplane, i.e., \( z^* = 0 \).

III. Governing Equations of Initial Functions

We assume that the top and bottom surfaces of a piezoelectric plate are subjected to given shear tractions \( q_{ij} \), normal tractions \(-q_{ij} \), and normal electric displacements \( D_3 \), which are scaled as

\[
\tau_{ij} = \varepsilon q_{ij}, \quad \varphi = -\varepsilon^2 q_{ij}, \quad D_3 = \varepsilon D_3^0
\]

(19)

Thus functions \( f \) and \( g \) in Eq. (17) are given by

\[
g^{(0)} = r^{(0)} = 0, \quad f^{(0)} = r^{(0)} = 0
\]

(20)

\[
g^{(n + 1)}(\pm a) = r^{(n + 1)}(\pm a) = q_{ij}, \quad f^{(n + 1)}(\pm a) = r^{(n + 1)}(\pm a) = -q_{ij}^2 \delta_{00}
\]

(21)
Using Eqs. (16) and (18) and \( \hat{\alpha}_3 \), Eq. (21) can be rewritten as

\[
\begin{align*}
\tau_{z_3}^{(n+1)}(0) + \hat{\alpha}_3 \tau_{z_3}^{(n)}(0) + \hat{\alpha}_3 \tau_{z_3}^{(n)}(0) & = \hat{\alpha}_3 \tau_{z_3}^{(n)}(0) + \hat{\alpha}_3 \tau_{z_3}^{(n)}(0) + \hat{\alpha}_3 \tau_{z_3}^{(n)}(0) \\
+ b^{(2)}(\pm a) \tau_{z_3}^{(n)}(0) & = \hat{\alpha}_3 \tau_{z_3}^{(n)}(0) + \hat{\alpha}_3 \tau_{z_3}^{(n)}(0) + \hat{\alpha}_3 \tau_{z_3}^{(n)}(0) \\
+ \sum_{k=1}^{n} \left[ b^{(k+1)}(\pm a) f_{L}^{(n-k)}(0) + b^{(k+2)}(\pm a) g_{L}^{(n-k)}(0) \right] & = \hat{\alpha}_3 \delta_{00} \\
\tau_{z_3}^{(n+1)}(0) & = \hat{\alpha}_3 \tau_{z_3}^{(n)}(0) + \hat{\alpha}_3 \tau_{z_3}^{(n)}(0) + \hat{\alpha}_3 \tau_{z_3}^{(n)}(0) + \hat{\alpha}_3 \tau_{z_3}^{(n)}(0) \\
+ b^{(2)}(\pm a) \tau_{z_3}^{(n)}(0) + \hat{\alpha}_3 \tau_{z_3}^{(n)}(0) + \hat{\alpha}_3 \tau_{z_3}^{(n)}(0) + \hat{\alpha}_3 \tau_{z_3}^{(n)}(0) \\
+ \sum_{k=1}^{n} \left[ b^{(k+1)}(\pm a) f_{L}^{(n-k)}(0) + b^{(k+2)}(\pm a) g_{L}^{(n-k)}(0) \right] & = -\hat{\alpha}_3 \delta_{00}
\end{align*}
\]

where an upper case subscript \( L \) takes values from 1 to 4 and

\[
U_{0}^{(n)} = \Phi_{0}^{(n)}(x_{r}, 0), \quad U_{1}^{(n)} = \Phi_{1}^{(n)}(x_{r}, 0), \quad \Phi_{0}^{(n)} = \Phi_{0}^{(n)}(x_{r}, 0)
\]

by eliminating four unknowns \( \tau_{z_3}^{(n+1)}(0), \tau_{z_3}^{(n+1)}(0), \) and \( D_{3}^{(n)}(0) \) from Eq. (22) and using the notation

\[
\begin{align*}
\hat{\alpha}^{(n)} & = \alpha^{(n)}(a) - \alpha^{(n)}(a), \quad \hat{\alpha}^{(n)} = \alpha^{(n)}(a) \\
\hat{\alpha}^{(n)} & = \alpha^{(n)}(a) - \alpha^{(n)}(a), \quad \hat{\alpha}^{(n)} = \alpha^{(n)}(a)
\end{align*}
\]

we obtain the following equations:

\[
\begin{align*}
\hat{\beta}^{(n)}_{32} \tau_{z_3}^{(n)}(0) + \hat{\beta}^{(n)}_{33} \tau_{z_3}^{(n)}(0) & = \hat{\beta}^{(n)}_{31} \tau_{z_3}^{(n)}(0) + \hat{\beta}^{(n)}_{32} \tau_{z_3}^{(n)}(0) \\
+ \sum_{k=1}^{n} \left[ b^{(k+1)}(\pm a) f_{L}^{(n-k)}(0) + b^{(k+2)}(\pm a) g_{L}^{(n-k)}(0) \right] & = \hat{\beta}^{(n)}_{30} \delta_{00} \\
\hat{\beta}^{(n)}_{30} & = \hat{\beta}^{(n)}_{31} g_{L}^{(n)}(0) + \hat{\beta}^{(n)}_{32} g_{L}^{(n)}(0) \\
+ \left[ \hat{\beta}^{(n)}_{32} \tau_{z_3}^{(n)}(0) + \hat{\beta}^{(n)}_{33} \tau_{z_3}^{(n)}(0) + \hat{\beta}^{(n)}_{31} \tau_{z_3}^{(n)}(0) + \hat{\beta}^{(n)}_{32} \tau_{z_3}^{(n)}(0) + \hat{\beta}^{(n)}_{33} \tau_{z_3}^{(n)}(0) + \hat{\beta}^{(n)}_{31} \tau_{z_3}^{(n)}(0) \right] & = \hat{\beta}^{(n)}_{30} \delta_{00} \\
\end{align*}
\]

for the four unknowns \( U_{0}^{(n)}, U_{1}^{(n)}, U_{2}^{(n)}, \) and \( U_{3}^{(n)} \). The right-hand sides of Eqs. (25) involve \( \tau_{z_3}^{(n)}(0), f_{L}^{(n)}(0), g_{L}^{(n)}(0), \) and \( g_{L}^{(n)}(0), (k = 0, \ldots, n - 1) \), which are determined from solutions of order up to \( (n - 1) \). The \( (n + 1)th \)-order out-of-plane stresses on the midplane, \( \tau_{z_3}^{(n+1)}(0) \) and \( \tau_{z_3}^{(n+1)}(0) \), can be solved from Eq. (22).

With the notations

\[
Q(\cdots) \equiv \int_{0}^{a} \cdots dz, \quad Q(\cdots) \equiv \int_{-a}^{a} \cdots dz
\]

the equations of leading order \((n = 0)\) can be written as

\[
CX^{(0)} = \Phi^{(0)}
\]

where the components of the differential operator \( C \) are

\[
\begin{align*}
C_{aa} & = -\hat{Q}L^{a}_{\rho_{y}} b_{\rho_{y}} b_{\rho_{y}} \delta_{a} \delta_{\rho_{y}} \delta_{\rho_{y}} \\
C_{ab} & = -\hat{Q}L^{a}_{\rho_{y}} b_{\rho_{y}} b_{\rho_{y}} \delta_{a} \delta_{\rho_{y}} \delta_{\rho_{y}} \\
C_{ba} & = -\hat{Q}L^{a}_{\rho_{y}} b_{\rho_{y}} b_{\rho_{y}} \delta_{a} \delta_{\rho_{y}} \delta_{\rho_{y}} \\
C_{bb} & = \hat{Q}L^{a}_{\rho_{y}} b_{\rho_{y}} b_{\rho_{y}} \delta_{a} \delta_{\rho_{y}} \delta_{\rho_{y}}
\end{align*}
\]

We notice that the electric potential \( \Phi^{(0)} \) can first be solved from Eq. (27), and the leading-order terms of the midplane displacements can then be solved from the remaining Eqs. (27) \( l = 1 \). If the plate is symmetric about its midplane, then \( C_{aa} \equiv C_{ab} \equiv C_{ba} \equiv 0, \) and Eq. (27) can be further simplified to decouple the in-plane displacements \( U_{0}^{(n)} \) from \( U_{1}^{(n)} \) and \( \Phi^{(0)} \). This special case has been studied.

For an elastic plate the differential operator matrix \( C \) given in Eq. (28) reduces to that of the classical plate theory for the bending of a thin monoclinic plate or laminate. For a piezoelectric plate Eq. (27) can be viewed as a set of classical equations governing the unknowns (29) under the action of surface tractions and electric displacements. The higher-order equations (25) show that all of the terms on the right-hand side are expressed as derivatives of the lower-order solutions. Accordingly, the preceding procedure may be used to generate an accurate three-dimensional solution provided that the solution of the two-dimensional classical plate equations has been found.

Furthermore, we note that the left-hand sides of Eqs. (25) of different orders contain the same coefficients, which, after a little simplification, turn out precisely to be the classical operators generalized to piezoelectricity. In principle, any numerical technique for solving the classical plate equations can simply be used to solve the three-dimensional problems for laminated piezoelectric plates.
IV. Example

We consider a piezoelectric strip \((|x_1| \leq a, \ x_2 \to \infty)\) and assume that the solution is independent of \(x_2\). The matrix operators (7) simplify to

\[
A = \begin{bmatrix} I & -J \delta_1^2 \\
- J^T \delta_1 & K \delta_1^2 \end{bmatrix}, \quad B = \begin{bmatrix} -L_{11} \delta_1^2 & -M \delta_1^2 \\
-M_{11}^T \delta_1 & N \end{bmatrix}
\]

and the field equations (27) for the leading order reduce to

\[
d_{13} U_{11}^{(0)} + d_{13} U_{111}^{(0)} + d_{13} U_{1111}^{(0)} = q_{a}^+ - q_{a}^- + Q M \delta_1^2 D_{11}^{*} \\
d_{13} U_{11}^{(1)} + d_{13} U_{111}^{(1)} + d_{13} U_{1111}^{(1)} = -(q_{a}^+ - q_{a}^-) + a(q_{a}^+ + q_{a}^-) + Q \delta_1^2 D_{1111}^{*}
\]

with

\[
m = -(3/d_{44}) Q \left[ J_{11}^2 (Q - \tilde{Q})(k_{a} - 4k_{3} \delta_{a} \alpha) L_{11}^{aa} - k_{11}^2 (Q - \tilde{Q})(k_{a} - 4k_{3} \delta_{a} \alpha) M_{11}^{aa} \right]
\]

Then Eq. (22) for \(n = 1\) gives

\[
D_{11}^{(1)}(0) = D_{11}^{*} \\
\equiv 6 \left( \frac{Q k_{11}^2}{d_{44}} \tilde{Q} - \tilde{Q} \right) J_{11}^2 (Q - \tilde{Q})(k_{a} - 4k_{3} \delta_{a} \alpha) L_{11}^{aa}
\]

The edge conditions (34) may be written in the form of their expansion coefficients as

\[
q_{a}^\pm = \left\{ \begin{array}{ll}
0, & \text{if } x_1 = \pm a \\
\bar{d}, & \text{if } x_1 = \pm a
\end{array} \right. \quad \varphi(\pm a) = 0
\]

The electric edge condition (34) implies that the edges are grounded at the midplane \(z = 0\). These are not pointwise conditions and therefore exclude the boundary-layer effect. The edge conditions (34) may be written in the form of their expansion coefficients as

\[
U_{11}^{(n)} = 0, \quad U_{111}^{(n)} = 0, \quad \delta_1 U_{1111}^{(n)} = 0, \quad \Phi_{111}^{(n)} = 0
\]

The solution satisfying the preceding edge conditions is

\[
U_{11}^{(n)} = k_{a} x_1 (x_1^2 - a^2), \quad U_{111}^{(n)} = k_{3} (x_1^2 - a^2)^2, \quad \Phi_{111}^{(n)} = l (x_1^2 - a^2)
\]

where

\[
k_{a} = \frac{-q_{a}^+ - q_{a}^-}{\tilde{d} a_{3}}, \quad k_{3} = \frac{-q_{a}^+ - q_{a}^-}{\tilde{d} a_{3}}
\]

The expansion series for the reference displacements terminates at their first term, while the expansion series for the reference electric potential terminates at its second term. Thus, the solution is complete, and the full through-thickness solution of all nonvanishing orders is obtained by substituting these reference values into Eq. (18):

\[
g_{11}^{(0)}(0) = u_{11}^{(0)}(0), \quad f_{11}^{(0)} = f_{11}^{(0)}(0) + a_{11}^{(1)} g_{11}^{(0)}(0), \quad \Phi_{111}^{(0)} = \Phi_{111}^{(1)}(0)
\]

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\]

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\]

where

\[
k_{a} = \frac{-q_{a}^+ - q_{a}^-}{\tilde{d} a_{3}}, \quad k_{3} = \frac{-q_{a}^+ - q_{a}^-}{\tilde{d} a_{3}}
\]

Here \(\tilde{d}\) is the inverse of the submatrix consisting of the first three rows and first three columns of \(d\). The solution of the other four unknowns in Eq. (22) for \(n = 0\) can then be obtained as

\[
\tau_{13}^{(0)}(0) = 6 \tilde{d} \left( k_{a} - 4k_{3} \delta_{a} \alpha \right) L_{11}^{aa} x_1 \\
\tau_{33}^{(0)}(0) = 0 \\
\tau_{33}^{(0)} = 0 \\
\tau_{33}^{(0)} = 0 \\
D_{11}^{(0)}(0) = D = D_{11}^{*} = 2t \tilde{d} K_{11}^{2}
\]

Equation (25), for \(n = 1\) can be rewritten as

\[
\alpha_{44}^{(1)}(0) = -\alpha_{44}^{(1)}(0) - \alpha_{44}^{(2)}(0) U_{11}^{(0)} - \alpha_{44}^{(3)}(0) U_{1111}^{(0)}
\]

whose solution is

\[
\Phi_{111}^{(1)} = m (x_1^2 - a^2)
\]
More explicitly, using Eqs. (36), (38), (40), and (42), the closed-form analytical solution in the interior of the plate is

\[ u_{\sigma} = -\varepsilon \left[ \frac{2}{s_1 \sigma_1} R_1 x_1 (x_1 - a^2) + 1 R_2 x_1 ^3 \right] + \varepsilon \left[ \frac{q_1 - \sigma_1}{R_1} R_2 x_1 \right] \]

\[ u_{\tau} = \kappa (x_1 - a^2)^2 + \varepsilon ^2 \left[ D R_1 x_1 + (q_2 - \sigma_2) R_2 x_1 (x_1 - a^2) + 1 R_2 x_1 \right] \]

\[ \varphi = \varepsilon \left[ s_1 (x_1 - a^2) + \frac{3}{9} R_1 x_1 ^3 \right] + \varepsilon ^3 \left[ s_1 R_1 ^2 + D R_1 x_1 + (q_2 - \sigma_2) R_1 x_1 ^2 + 2 R_2 x_1 \right] \]

\[ \theta_{\omega} = \varepsilon (q_1 - \sigma_1) R_2 x_1 \]

\[ \theta_{\omega} = \varepsilon ^i \left[ 3 \sigma_1 - (q_2 - \sigma_2) R_1 \right] \]

(47)

(48)

(49)

The in-plane stress field and the in-plane electric displacement components, which may be discontinuous across interfaces, are determined from Eq. (11) to be

\[ \tau_{\omega} = e \left[ a \left( M_4 \sigma_4 + \frac{1}{2} a_4 \sigma_4 + a_4 \right) + \frac{1}{2} a_4 \sigma_4 \right] \]

\[ - (q_2 - \sigma_2) L_{\sigma_2} R_2 \left( 3 x_1 ^2 - a^2 \right) + \varepsilon \left[ s_1 M_4 + D' M_2 a \right] \]

\[ - (q_2 - \sigma_2) \left[ (M_4 + L_{\sigma_4} R_2) - L_{\sigma_4} R_2 \right] + \frac{2 R_2}{1 + R_2} x_1 \]

\[ D_{\sigma} = -2 \frac{R_2}{1 + R_2} x_1 + \varepsilon ^i \left[ (q_1 - \sigma_1) \left( L_{\sigma_2} R_2 - 2 K_{\sigma_2} R_5 \right) + 2 m R_5 \right] \]

(50)

Thus the exact closed-form solution of the problem has been obtained.

By setting the piezoelectric moduli \( q_{ij} = 0 \), we obtain results for a purely elastic strip that are slightly different from the solution of Ref. 21 because of the different choice of the reference planes. For a piezoelectric strip symmetric about the midplane, it can be verified that the present solution is exactly the same as that given in Ref. 12, where a symmetric elliptic plate is degenerated into an infinite strip.

The transverse shear stresses \( \tau_{\omega} \) given by Eq. (46) are independent of the relevant transverse shear moduli \( c_{ijkl} \), a property already known for elliptic monoclinic elastic plates. Here this property is shown to be valid for monoclinic piezoelectric strips. In addition, the out-of-plane stresses \( \tau_{\omega} \) and \( \theta_{\omega} \) in Eq. (47) do not depend on the normal electric displacement loading \( D_1 \), prescribed on the top and bottom surfaces of the plate. This means that the specified uniform electric displacements do not produce any out-of-plane stresses, another property particularly valid for the piezoelectric strips.

Differentiation of \( \varphi \) in Eq. (48) with respect to \( x_1 \) and \( x_2 \) yields the following order of magnitude estimates of the electric field components:

\[ E_1 = -\varphi_1 \sim (D_1 - D_2) (O) \]

\[ \frac{q_2 - \sigma_2}{R_1} \]

\[ \frac{q_2 - \sigma_2}{R_1} \]

\[ \frac{q_2 - \sigma_2}{R_1} \]

\[ \frac{q_2 - \sigma_2}{R_1} \]

(51)

Thus, if \( D_1 - D_2 = 0 \), the ratio of the in-plane to the out-of-plane electric field components is of the order of the dimensionless thickness parameter, i.e., \( E_1 / E_2 \sim \varepsilon \), which implies that for a thin piezoelectric plate \( E_1 \) is negligibly small as compared to \( E_2 \). On the other hand, \( E_1 \) should not be neglected when \( D_1 - D_2 \neq 0 \) because it is of the order of the reciprocal of the thickness parameter, i.e., \( E_1 / E_2 \sim 1 / \varepsilon \). In such a case the in-plane electric field component is in fact much larger than the out-of-plane component. Therefore the assumption that the in-plane electric field component is negligible is incorrect when the transverse electric displacements on the top and bottom surfaces are unequal to each other. However, for equal transverse electric displacements prescribed on the top and bottom surfaces, the existing thin piezoelectric plate models based on negligible in-plane electric field components are satisfactory in this aspect. This observation from the solution of laminated piezoelectric strips of arbitrary thickness agrees with that derived from the limit analysis of thin single-layer plates.

For a piecewise homogeneous piezoelectric strip all of the material moduli are piecewise constants in the thickness direction. Observing Eqs. (46–50), the in-plane and out-of-plane distributions of the expansion terms of the mechanical displacements, stresses, electric displacements, and electric potential are summarized in Table 1. Of particular interest are the through-thickness distributions of the physical quantities. The through-thickness distributions of the expansion terms of the mechanical and electric quantities in \( z \) and \( \varepsilon \) are of the same form. Table 1 provides useful information for making proper approximations in two-dimensional piezoelectric plate models. In particular, the assumption of the through-thickness cubic distribution of the in-plane mechanical displacements in higher-order plate theories agrees with the present results. However, the constant distribution of the out-of-plane mechanical displacement provides only the zeroth-order approximation to our solution.

### Table 1

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( x_1 )</th>
<th>Constant</th>
<th>Linear</th>
<th>Quadratic</th>
<th>Cubic</th>
<th>Quartic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau_{13} )</td>
<td>( D_{13} )</td>
<td>( \varphi_{13} )</td>
<td>( \psi_{13} )</td>
<td>( \theta_{13} )</td>
<td>( \omega_{13} )</td>
<td>( \eta_{13} )</td>
</tr>
</tbody>
</table>

In the above table, \( D_{13} \) is defined as the linear expansion term of the electric displacement in the thickness direction. \( \varphi_{13} \) and \( \psi_{13} \) are the linear and quadratic terms of the in-plane electric displacement, respectively. \( \theta_{13} \) and \( \omega_{13} \) are the linear and quadratic terms of the through-thickness electric potential, respectively. \( \eta_{13} \) is the constant term of the through-thickness electric potential.

V. Numerical Results

The materials used to compute numerical results in the following examples are lead zirconate titanate (PZT-4) and polyvinylidene fluoride (PVDF). The material moduli of PZT-4 and PVDF are given in Table 2, where \( \varepsilon_0 \) is the permittivity of vacuum. Because a linear theory is used, results for complex loadings can be obtained by a superposition of the results for simple loadings. Two
loading conditions are examined. One corresponds to applied uniform normal traction with vanishing normal electric displacements and the other to applied uniform normal electric displacements with vanishing normal pressures. Results are presented in terms of the nondimensional variables

\[
\begin{align*}
\bar{a}_i &= u_i / Pa, \\
\tau_{ij} &= \tau_{ij} / Pe^*, \\
\phi &= e^* \phi / Pae^*, \\
D_i &= D_i / Pe^*
\end{align*}
\]

with \( e^* = 10^9 \text{N/m}^2 \), \( \epsilon^* = 1 \text{C/m}^2 \), and either \( P = q_3 \) for applied normal traction \( q_3 \) or \( P = D_3 \) for applied normal electric displacement \( D_3 \).

A four-ply (PZT-4/90-deg PVDF/PZT-4/0-deg PVDF) laminate is considered and the tests of the laminate plate are listed in Table 3. The through-thickness distributions of the transverse shear stress \( \tau_{33} \) and the transverse normal stress \( \tau_{33} \) are shown in Figs. 1 and 2.

To illustrate the order of magnitude of the electric field components, we consider a homogeneous single-layer PZT-4 plate with normal electric displacements \( D_3^* \) prescribed on its top and bottom surfaces. For symmetric electric loading \( D_3^* = D_3 \), the ratio of the in-plane to the out-of-plane electric field components \( E_i/E_3 \) can be shown to be identically zero. For antisymmetric electric loading \( D_3^* = -D_3^* \), we have plotted in Fig. 3 the through-thickness distribution of \( E_i/E_3 = \beta x_i / a \) within the lower half plate, as the result for the upper half plate is antisymmetric about the midplane. Here

\[
\beta = 2a / \epsilon (M_{12}^2 R_0^* + N_{22} R_0^*)
\]

is a function of \( x_i \). Because \( E_i = 0 \) at the midplane of the plate, the value of \( \beta \) approaches infinity as \( x_i \to 0 \) and thus is not shown therein for \( -0.25 < x_i / h < 0 \). Figure 3 shows that for this case the in-plane electric field component cannot be neglected as compared with the out-of-plane electric field component.

---

**Table 2** Values of nonvanishing material moduli for PZT-4 and PVDF

<table>
<thead>
<tr>
<th>Moduli</th>
<th>PZT-4</th>
<th>PVDF</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_{1111} ), GPa</td>
<td>139</td>
<td>238.24</td>
</tr>
<tr>
<td>( c_{2222} ), GPa</td>
<td>139</td>
<td>23.6</td>
</tr>
<tr>
<td>( c_{3333} ), GPa</td>
<td>115</td>
<td>10.64</td>
</tr>
<tr>
<td>( c_{1122} ), GPa</td>
<td>77.8</td>
<td>3.98</td>
</tr>
<tr>
<td>( c_{1313} ), GPa</td>
<td>74.3</td>
<td>2.19</td>
</tr>
<tr>
<td>( c_{2323} ), GPa</td>
<td>74.3</td>
<td>1.92</td>
</tr>
<tr>
<td>( c_{3333} ), GPa</td>
<td>25.6</td>
<td>2.15</td>
</tr>
<tr>
<td>( c_{1111} ), GPa</td>
<td>25.6</td>
<td>4.4</td>
</tr>
<tr>
<td>( c_{1212} ), GPa</td>
<td>30.6</td>
<td>6.43</td>
</tr>
<tr>
<td>( \epsilon_{311} ), ( \text{C/m}^2 )</td>
<td>-5.2</td>
<td>-0.13</td>
</tr>
<tr>
<td>( \epsilon_{232} ), ( \text{C/m}^2 )</td>
<td>-5.2</td>
<td>-0.145</td>
</tr>
<tr>
<td>( \epsilon_{333} ), ( \text{C/m}^2 )</td>
<td>15.1</td>
<td>-0.276</td>
</tr>
<tr>
<td>( \epsilon_{222} ), ( \text{C/m}^2 )</td>
<td>12.7</td>
<td>-0.009</td>
</tr>
<tr>
<td>( \epsilon_{133} ), ( \text{C/m}^2 )</td>
<td>12.7</td>
<td>-0.135</td>
</tr>
<tr>
<td>( \epsilon_{111} ), ( \text{C/m}^2 )</td>
<td>1475</td>
<td>12.5</td>
</tr>
<tr>
<td>( \epsilon_{223} ), ( \text{C/m}^2 )</td>
<td>1475</td>
<td>11.98</td>
</tr>
<tr>
<td>( \epsilon_{333} ), ( \text{C/m}^2 )</td>
<td>1300</td>
<td>11.98</td>
</tr>
</tbody>
</table>

\( a_{0} = 0.0584185 \text{pF/m} \).

**Table 3** Results at some points for a clamped-clamped four-ply (PZT-4/90-deg PVDF/PZT-4/0-deg PVDF) laminate (2a/h = 10)

<table>
<thead>
<tr>
<th>Parameter variation</th>
<th>Mechanical load ( q_3^* )</th>
<th>Electric load ( D_3^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_0(h/2) )</td>
<td>0.02485</td>
<td>-0.005980</td>
</tr>
<tr>
<td>( a_0(h/2) )</td>
<td>0.4905</td>
<td>-0.01078</td>
</tr>
<tr>
<td>( D_1(0, h/2) )</td>
<td>0.6177</td>
<td>-4.972</td>
</tr>
<tr>
<td>( D_0(0, 0) )</td>
<td>-0.01637</td>
<td>0.4999</td>
</tr>
<tr>
<td>( \tau_{11}(0, (h/4)^+ )</td>
<td>4.319</td>
<td>-0.2695</td>
</tr>
<tr>
<td>( \tau_{22}(0, (h/4)^+ )</td>
<td>2.099</td>
<td>0.5102</td>
</tr>
<tr>
<td>( \phi(0, 0) )</td>
<td>0.07356</td>
<td>-0.2565</td>
</tr>
</tbody>
</table>

Fig. 1 Through-the-thickness distribution of the dimensionless transverse shear stress for a four-ply (PZT-4/90-deg PVDF/PZT-4/0-deg PVDF) laminate under \( q_3^* \) (2a/h = 10).

Fig. 2 Through-the-thickness distribution of the dimensionless transverse normal stress for a four-ply (PZT-4/90-deg PVDF/PZT-4/0-deg PVDF) laminate under \( q_3^* \) (2a/h = 10).

Fig. 3 Through-the-thickness distribution of the ratio of the in-plane to the out-of-plane electric field components for a homogeneous PZT-4 plate under \( D_3^* = -D_3^* \) (2a/h = 10).
Although the numerical results for only the span-to-thickness ratio $2\alpha h = 10$ are given, many other results can simply be obtained in terms of expressions of physical quantities and Table 1. For example, for $2\alpha h = 100$ the value of $\beta$ will be 10 times the value given in Fig. 3 for $2\alpha h = 10$ according to Eq. (53), and $\tau_{\alpha 2}$ will be the same at a plane parallel to the midplane according to Table 1.

VI. Conclusions

We have combined the method of asymptotic series expansion with the transfer matrix method to obtain a closed-form solution for the three-dimensional electromechanical deformations of a clamped-clamped laminated plate. The governing equations are expressed in terms of functions defined on the midplane of the plate. Equations for the determination of the $n$th-order mechanical displacements and electric potentials involve effective loads that depend upon the quantities of order 0 through $(n - 1)$. For a clamped-clamped piezoelectric strip of length $2\alpha$, these are further simplified, and a closed-form solution of the governing equations is derived. The transverse shear stresses $\tau_{\alpha 2}$ ($\alpha = 1, 2$) are independent of the relevant transverse shear moduli $c_{\omega\alpha\beta}$, and $\tau_{\alpha 2}$ and the transverse normal stress $\tau_{\alpha 3}$ do not depend upon the uniform normal electric displacement $D_3^e$ prescribed on the top and bottom surfaces of the laminate. For $D_3^e = D_3^o$, the ratio of the in-plane to the out-of-plane electric field components is of the order of $h/2\alpha a$, where $h$ is the thickness of the laminate. However, if $D_3^e \neq D_3^o \neq 0$, then the in-plane electric field is much higher than the out-of-plane electric field and cannot be neglected as is often done in some plate theories. Guidelines for making proper approximations for the mechanical and electric quantities in two-dimensional plate theories are provided in Table 1. In particular, the assumption of the through-thickness cubic variation of the in-plane mechanical displacements agrees with the closed-form solution derived here for the cylindrical bending deformations of a piezoelectric laminate.

Numerical results are presented for a piezoelectric laminated strip subjected to either mechanical or electric loads on the top and bottom surfaces.

The present solution could serve for checking the validity of other approximate two-dimensional theories and numerical methods.

Acknowledgments

This work was partially supported by Army Research Office Grant DAAG55-98-1-0030 to National Science Foundation Grant CMS9713453 to Virginia Polytechnic Institute and State University.

References


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Associate Editor