Second-order constitutive relations for transversely isotropic piezoelectric porous materials

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Based on the theory of invariants, polynomial constitutive relations for transversely isotropic piezoelectric porous materials are derived from the polynomial integrity bases for an energy density function depending on a symmetric second-order tensor and two vectors. They are assumed to be smooth functions of their arguments, are expanded about the values their arguments take in the reference configuration and all terms up to the quadratic terms in the gradients of the mechanical displacement, the electric potential, and the gradients of the volume fraction are kept. The second-order constitutive relations so obtained are then specialized to the case of infinitesimal deformations and weak electric fields, and also to the case of infinitesimal deformations and strong electric fields.

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INTRODUCTION

The effect of nonlinearity in the constitutive relations of piezoelectric ceramics has been of recent interest because of their use in smart structures. Nelson has given, for all crystal classes, representations of quadratic piezoelectric constitutive relations generated by an energy density function of a symmetric second-order tensor and a vector.

Many piezoelectric materials are porous. Here, based on the theory of invariants, nonlinear form invariant polynomial constitutive relations for transversely isotropic piezoelectric porous materials are derived. They are then reduced to second-order and linear constitutive relations, and constitutive relations for small deformations and strong electric fields.

I. EQUATIONS FOR A NONLINEAR PIEZOELECTRIC POROUS MATERIAL

Let the coordinates of a material particle with respect to a rectangular Cartesian coordinate system be \( x_K \) in the reference configuration, its spatial coordinates in the current configuration be \( x_k \), then the balance laws for a nonlinear piezoelectric porous material are

\[
\begin{align*}
\rho_0 = \rho \mathcal{Z}, & \quad \mathcal{Z} = \det(x_k, x_K), \\
\mathbf{T}_{KL} x_{KL} + \mathcal{J} x_K x_K \mathbf{e}_0 (E_K E_I - \frac{1}{2} \mathbf{E}_m \mathbf{E}_m \delta_{KL}) & = \rho_0 \delta_{KK} \phi, \\
(\Pi_K - \mathcal{J} x_K x_K \mathbf{e}_0 \mathbf{e}_0) & = 0, \\
h_{KK} + \rho_0 (I + g) & = \rho_0 (k \psi),
\end{align*}
\]

where \( \rho \) is the mass density, \( \rho_0 \) is the mass density of the porous material in the reference configuration, \( T_{KL} \) is the second Piola-Kirchhoff stress tensor, \( \mathcal{Z} \) is the mechanical displacement vector, \( \delta_{KK} \) is the shifter, \( \Pi_K \) is the material electric polarization, \( E_k = -\phi_k \) is the electric field, \( \phi \) is electric potential, \( \mathbf{e}_0 \) is the permittivity of the free space, \( \delta \) is the Kronecker delta, \( h_K \) is the equilibrated stress, \( g \) is the intrinsic equilibrated body force, \( I \) is the extrinsic equilibrated body force, \( k \) is the equilibrated inertia, and \( \psi \) is the volume fraction of voids or the porosity of the material. Throughout this paper, a repeated index implies summation over the range of the index, and a comma followed by \( \mathcal{Z}_0 \) implies partial differentiation with respect to \( x_k \). A dot above a quantity signifies its material time derivative. Balance laws (1) are accompanied by constitutive relations

\[
\mathbf{T}_{KL} = \frac{\partial \Sigma}{\partial E_{KL}}, \quad \Pi_K - \mathcal{J} x_K x_K \mathbf{e}_0 \mathbf{e}_0 = \frac{\partial \Sigma}{\partial W_K}, \quad h_{KK} = \frac{\partial \Sigma}{\partial V_K},
\]

where \( \Sigma(E_{KL}, W_K, V_K) \) is an energy density function that also depends on \( \psi \); this dependence is not written explicitly. In Eq. (2) \( E_{KL} \) is the Green–Lagrange strain tensor, \( W_K \) is the electric field in material form, and \( V_K \) is the material gradient of \( \psi \).

II. FORM INVARIANT POLYNOMIAL CONSTITUTIVE RELATIONS

Let the material be invariant under rotations about a unit vector \( a \) and reflections about planes containing \( a \). Then any scalar polynomial function of a symmetric tensor \( E \) and two vectors \( W \) and \( V \) must be a polynomial function of the following invariants called the polynomial integrity bases:

\[
I_1 = a \cdot E \cdot a, \quad I_2 = tr E, \quad I_3 = a \cdot W, \quad I_4 = a \cdot V,
\]

\[
II_1 = a \cdot E^2 \cdot a, \quad II_2 = tr E^2, \quad II_3 = W \cdot W, \quad II_4 = a \cdot W \cdot W \cdot a, \quad II_5 = V \cdot V,
\]

\[
III_1 = a \cdot E \cdot V + V \cdot E \cdot a, \quad III_2 = V \cdot W, \quad III_3 = W \cdot V,
\]
\[ T = \sum_{i} \frac{\partial}{\partial \mathbf{E}_i} \left( \sum_{j} \frac{\partial}{\partial \mathbf{W}_j} \left( \sum_{k} \frac{\partial}{\partial \mathbf{V}_k} \mathbf{F} \right) \right) \]

where \( \mathbf{E}_i \), \( \mathbf{W}_j \), and \( \mathbf{V}_k \) are the symmetric tensor \( \mathbf{E} \) and vectors \( \mathbf{W} \) and \( \mathbf{V} \), respectively.

\[ \Sigma = \sum (I_1, \ldots, I_{14}, I_{15}, \ldots, I_{17}, I_{18}, \ldots, I_{36}) \]

where \( \Sigma \) is a general polynomial function of its arguments. From (2) and (5), we obtain the following general form for the polynomial constitutive relations for a nonlinear transversely isotropic material:

\[ T = a_{\alpha} a_{\beta} + 2 \alpha \beta + \frac{\partial}{\partial \mathbf{E}_i} \left( \sum_{j} \frac{\partial}{\partial \mathbf{W}_j} \left( \sum_{k} \frac{\partial}{\partial \mathbf{V}_k} \mathbf{F} \right) \right) \]

where \( I \) is the identity tensor and \( \mathbf{u} \otimes \mathbf{v} \) denotes tensor product between tensors \( \mathbf{u} \) and \( \mathbf{v} \). In order to derive a second-order theory, we assume that \( \Sigma \) is a smooth function of its arguments and write its Taylor series expansion about the values the arguments take in the reference configuration and only keep all terms up to degree three in \( \mathbf{E} \), \( \mathbf{W} \), and \( \mathbf{V} \):

\[ \Sigma = \sum_{i} \frac{\partial}{\partial \mathbf{E}_i} \left( \sum_{j} \frac{\partial}{\partial \mathbf{W}_j} \left( \sum_{k} \frac{\partial}{\partial \mathbf{V}_k} \mathbf{F} \right) \right) \]

where \( \alpha \), \( \beta \), and \( \gamma \) will be shown to represent initial fields. Here, \( c_1 - c_4 \), \( e_1 \), \( e_2 \), \( \kappa_1 \), \( \kappa_2 \), \( e_1 - e_3 \), \( f_1 - f_3 \), \( g_1 \), and \( g_2 \) are 17 constants for the quadratic terms in \( \Sigma \) or for the linear constitutive relations; \( \lambda_1 - \lambda_9 \) represent cubic terms containing \( \mathbf{E} \) alone, \( \mu_1 \) and \( \nu_2 \) cubic terms containing \( \mathbf{W} \) alone, \( \xi_1 \) and \( \xi_2 \) cubic terms containing \( \mathbf{V} \) alone, \( \nu_1 \) - \( \nu_6 \) cubic terms containing \( \mathbf{E} \) and \( \mathbf{W} \), \( \eta_1 - \eta_6 \) cubic terms containing \( \mathbf{E} \) and \( \mathbf{V} \), \( \zeta_1 - \zeta_6 \) cubic terms containing \( \mathbf{W} \) and \( \mathbf{V} \), and \( \delta_1 - \delta_6 \) cubic terms containing all three tensors. Substitution of (9) into (6) - (8) gives

\[ \Sigma = \sum (I_1, I_2, \ldots, I_{17}, I_{18}, \ldots, I_{36}) \]

where \( \sum \) is a general polynomial function of its arguments. From (2) and (5), we obtain the following general form for the polynomial constitutive relations for a nonlinear transversely isotropic material:

\[ T = \sum (I_1, I_2, I_3, \ldots, I_{36}) \]

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Equations (10)–(12) are polynomial representations of $T$, $\Pi$, and $h$ of degree two in components of $E$, $W$, and $V$. It can be seen that terms in (10) involving $\alpha_1$ and $\alpha_2$ do not depend upon $E$, $W$, or $V$ and hence represent the initial stress. Terms involving $\beta$ in (11) and $\gamma$ in (12) are similar. The derivations in this section are for the constitutive relations generated by an energy density function of a symmetric tensor and two vectors in general. When applied to the case of a piezoelectric porous material, all of the material parameters should be considered as functions of $\psi$, the porosity of the material.

Passman and Batra\(^6\) assumed that $T$, $\Pi$, and $h$ also depend upon $\dot{\psi}$, the rate of change of the porosity. If we adopt this assumption, then the material parameters will also depend upon $\dot{\psi}$. Henceforth, we disregard the dependence of $T$, $\Pi$, and $h$ upon $\dot{\psi}$.

### III. SECOND-ORDER CONSTITUTIVE RELATIONS

By second-order constitutive relations we mean relations that contain all quadratic terms of the mechanical displacement gradient, electric potential gradient, and the gradient of the volume fraction of voids. Equations (10)–(12) contain some higher-order terms in this sense. To get second-order constitutive relations, we make the following decompositions:

$$E = E^{(1)} + E^{(2)}, \quad E^{(1)}_{KL} = \frac{1}{2}(U_{KL} + U_{LK}),$$

where $E^{(2)}$ is a second-order tensor.

$$\Pi = \Pi^{(1)} + \Pi^{(2)},$$

$$h = h^{(1)} + h^{(2)}.$$
IV. LINEAR CONSTITUTIVE RELATIONS

The linear or first-order constitutive relations (16), (18), and (20), when terms are rearranged according to the order of dependence on $E^{(1)}$, $W^{(1)}$, and $V^{(1)}$, are

$$
    T^{(1)} = \left( 2c_3 l_1^{(1)} + c_3 l_1^{(1)} \right) a \otimes a + \left( 2c_2 l_2^{(1)} + c_1 l_3^{(1)} \right) l + c_4 (a \otimes E^{(1)} \cdot a + a \cdot E^{(1)} \otimes a) + 2c_5 E^{(1)} + e_1 l_3^{(1)} a \otimes a + e_2 l_1^{(1)} l + e_3 (a \otimes W^{(1)} + W^{(1)} \otimes a) + f_1 l_4^{(1)} a \otimes a + f_2 l_4^{(1)} l + f_3 (a \otimes V^{(1)} + V^{(1)} \otimes a),
$$

$$
    \Pi^{(1)} = -\left( e_1 l_1^{(1)} + e_2 l_2^{(1)} a - 2e_3 E^{(1)} \cdot a - 2e_4 l_3^{(1)} a - 2e_5 W^{(1)} - g_1 l_4^{(1)} a - g_2 V^{(1)}
\right),
$$

$$
    h^{(1)} = \left( f_1 l_1^{(1)} + f_2 l_2^{(1)} a + 2f_3 E^{(1)} \cdot a + g_1 l_3^{(1)} a + g_2 W^{(1)} + 2 \kappa_1 l_4^{(1)} a + 2 \kappa_2 V^{(1)}.
\right)
$$

These can be written in a matrix form.\(^8\)\(^9\)

V. SMALL DEFORMATIONS AND STRONG ELECTRIC FIELDS

In this case, (15), (16), (18), and (20) remain the same. Equations (17), (19), and (21) reduce to

$$
    T^{(2)} = \left( \nu_1 l_1^{(2)} + \nu_2 l_2^{(2)} \right) a \otimes a + \left( \nu_3 l_3^{(2)} \right) l + \nu_4 (a \otimes E^{(2)} \cdot a + a \cdot E^{(2)} \otimes a) + 2 \nu_5 E^{(2)} + \nu_6 l_3^{(2)} l + \nu_7 (a \otimes W^{(2)} + W^{(2)} \otimes a) + \nu_8 W^{(2)} \otimes W^{(2)} + \nu_9 W^{(2)} \otimes W^{(2)},
$$

$$
    \Pi^{(2)} = -\left( 3 \mu_1 l_1^{(2)} + \mu_2 l_3^{(2)} a - 2 \mu_3 l_3^{(2)} W^{(2)}
\right),
$$

$$
    h^{(2)} = \left[ \xi_1 (l_1^{(2)})^2 + \xi_2 l_3^{(2)} a + \xi_3 l_3^{(2)} W^{(2)}
\right],
$$

which can also be written in a matrix form.\(^8\)\(^9\)

VI. CONCLUSIONS

Constitutive relations (22) imply that for a linear theory, contributions to stresses, electric polarization, and the equilibrated stresses from the mechanical strains, the electric field, and the porosity gradients are additive. However, such is not the case in the nonlinear theory as should be evident from the second-order constitutive relations (15)–(21). When the changes in porosity are also infinitesimal, then the above-stated constitutive relations can be suitably modified. In them, the material parameters will depend upon the initial value of the porosity.

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8. R. C. Batra and J. S. Young, "Second-order constitutive relations for 'transversely isotropic piezoelectric porous materials'" (available from the authors).