Analysis of Adiabatic Shear Bands by Numerical Methods

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Nomenclature

\( \rho \) mass density in the present configuration
\( \rho_0 \) mass density in the reference configuration
\( J \) determinant of the deformation gradient
\( F_{ia} \) components of deformation gradient
\( x_i \) coordinates of a point in the present configuration
\( X_\alpha \) coordinates of a point in the reference configuration
\( T_{ia} \) components of the first Piola–Kirchhoff stress tensor
\( v_i \) components of velocity
\( b_i \) components of body force per unit mass
\( e \) specific internal energy
\( Q_\alpha \) components of heat flux per unit area in the reference configuration
\( r \) supply of internal energy per unit mass
\( \Omega \) region occupied by the body in the reference configuration
\( \partial \Omega \) boundary of \( \Omega \)
\( \theta \) temperature of a material point
\( N_\alpha \) components of the outward unit normal to the boundary
\( f_i \) components of the present force per unit undeformed area
\( t \) time
\( \phi \) empty set
\( h \) prescribed heat flux per unit undeformed area
\( \dot{\theta} \) time rate of change of \( \theta \)
\( c \) specific heat
\( \phi_i \) components of test function
\( \eta \) scalar test function
\( H^1 \) collection of real valued functions whose first-order derivative is square integrable
\( S_m \otimes S_\theta \) tensor product between sets \( S_m \) and \( S_\theta \)
\( \psi_\Lambda \) finite element basis function for node A for displacement degree of freedom
\( \zeta_\Lambda \) finite element basis function for node A for temperature degree of freedom
\( d_\Lambda \) displacement of node A along \( x_r \)-axis
\( \theta_\Lambda \) temperature of node A
4.1 Introduction

Other chapters in this volume describe experimental and analytical works on adiabatic shear banding, and constitutive relations for a microporous thermo-elasto-viscoplastic material. No attempt is made here to describe all numerical methods that have been used to delineate adiabatic shear bands (ASBs). Rather, the focus is on illuminating a numerical technique, namely the finite element method (FEM), commonly used to analyse initial-boundary-value problems (IVBPs) for non-linear thermo-elasto-viscoplastic problems, and illustrating some results obtained using this technique. The interested reader is referred to books [1—4] and review articles.
[5–7] for studying earlier works on ASBs, and to books [8–10] for studying the FEM. Suffice it to say that the phenomenon was discovered experimentally by Tresca [11] in 1878 and by Massey [12] in 1921 during high strain-rate compressive deformations of a steel block. Kalthoff [13–15] observed ASBs in a rectangular plate with two identical prenotches parallel to the short side and impacted on the notched side by a circular cylindrical projectile made of the same material as the plate with the projectile diameter equal to the distance between the two notches. For low-impact speeds, a crack initiated from the notch tip indicative of brittle failure, but at high-impact speeds an ASB initiated; the transition in the failure mode also depended on the notch-tip radius and strain hardening, strain-rate hardening and thermal-softening characteristics of the plate material. This problem has been numerically studied by Zhou et al. [16,17], Batra and co-workers [18–21] as well as Wang and Liu [22] who have cited several other works. However, work related to the Kalthoff problem is not discussed here because of space limitations.

An ASB usually precedes shear fracture in ductile materials. Whereas it is desirable to delay the initiation and growth of ASBs in most structures, their occurrences in high kinetic energy penetrators are promoted because it is believed that they enhance the penetration of the rod into a target by continuously making the penetrator conically nose shaped.

4.2 Brief Overview of the FEM

Equations, in the referential description of motion, governing thermo-mechanical deformations of a body are the following:

\[ \rho J = \rho_0, \quad \text{in } \Omega \]  \hspace{1cm} (4.1)

\[ \rho_0 \dot{v}_i = T_{i\alpha} + \rho_0 b_t \quad \text{in } \Omega \]  \hspace{1cm} (4.2a)

\[ T_{i\alpha} F_{j\alpha} = T_{j\alpha} F_{i\alpha} \quad \text{in } \Omega \]  \hspace{1cm} (4.2b)

\[ \rho_0 \dot{e} = -Q_{\alpha,\alpha} + T_{i\alpha} F_{i\alpha} + \rho_0 r \quad \text{in } \Omega \]  \hspace{1cm} (4.3)

Here, \( \rho \) is the present mass density of a material particle whose mass density in the reference configuration is \( \rho_0 \), \( J = \det[F_{i\alpha}] \) is the Jacobian, \( F_{i\alpha} = \partial x_i / \partial X_\alpha \), \( x_i \) is the present position of the material particle that occupied place \( X_\alpha \) in the reference configuration, \( T_{i\alpha} \) is the first Piola–Kirchhoff stress tensor, \( v_i \) is the velocity, a superimposed dot indicates the material time derivative, \( b_t \) is the body force per unit mass, a comma followed by \( \alpha \) denotes the partial derivative with respect to \( X_\alpha \), \( e \) is the specific internal energy, \( Q_\alpha \) is the heat flux per unit reference area, \( r \) is the supply of internal energy per unit mass and a repeated index implies summation over the ranges 1, 2 and 3 of the index. Equations (4.1)–(4.3) are written in rectangular
Cartesian coordinates, and \( \Omega \) is the region occupied by the body in the reference configuration. These equations are supplemented by constitutive relations that characterize the material of the body, and initial and boundary conditions such as those listed here:

\[
x_i(X_\alpha, t) = \bar{x}_i, \quad X_\alpha \in \partial \Omega
\]

\[
\theta(X_\alpha, t) = \bar{\theta}, \quad X_\alpha \in \partial \Omega
\]

\[
T_{i\alpha}(X_\beta, t)N_\alpha = f_i, \quad X_\beta \in \partial \Omega
\]

\[
- Q_{\alpha}(X_\beta, t)N_\alpha = h, \quad X_\beta \in \partial \Omega
\]

\[
\partial_1 \Omega \cap \partial_2 \Omega = \phi, \quad \partial_1 \Omega \cup \partial_2 \Omega = \partial \Omega
\]

\[
\partial_3 \Omega \cap \partial_4 \Omega = \phi, \quad \partial_3 \Omega \cup \partial_4 \Omega = \partial \Omega
\]

\[
x_i(X_\alpha, 0) = x_i^0(X_\alpha), \quad \dot{x}_i(X_\alpha, 0) = \dot{x}_i^0(X_\alpha), \quad \theta(X_\alpha, 0) = \theta^0(X_\alpha)
\]

Boundary conditions (4.4) and (4.5) are usually called essential, and Eqs (4.6) and (4.7) natural. In them, \( N_\alpha \) is an outward unit normal to the boundary \( \partial \Omega \) of \( \Omega \) in the reference configuration, and functions \( x_i^0, \theta^0, \dot{x}_i, \bar{x}_i, \bar{\theta}, f_i \) and \( h \) are presumed to be given. We will postulate constitutive relations later, and merely state for the time being that \( T_{i\alpha}, Q_{\alpha}, \) and \( e \) may be functions of time histories of \( F_{i\alpha}, \dot{F}_{i\alpha}, \theta \) and \( \theta_{i\alpha} \) with \( \theta \) being the absolute temperature of a material particle. The constitutive relation for \( T_{i\alpha} \) is such that the balance of the moment of momentum (4.2b) is identically satisfied; it is thus not considered hereafter. Equation (4.3) can alternatively be written as:

\[
\rho_0 \dot{\theta} = - Q_{\alpha,\alpha} + T_{i\alpha}^{\text{ne}} \dot{F}_{i\alpha} + \rho_0 c \dot{\theta}
\]

where \( T_{i\alpha}^{\text{ne}} \) is the non-equilibrium part of the first Piola–Kirchhoff stress tensor,

\[
T_{i\alpha}^{\text{ne}} = T_{i\alpha} - T_{i\alpha}(\mathbf{F}, 0, \theta, 0), \quad \dot{e} = T_{i\alpha}(\mathbf{F}, 0, \theta, 0) \dot{F}_{i\alpha} + c \dot{\theta}
\]

and \( c \) is the specific heat, which may depend upon \( \mathbf{F} \) and \( \theta \). Thus, Eqs (4.1)–(4.3) involve second-order derivatives with respect to \( X_\alpha \) of unknown functions \( x_i(X_\alpha, t) \) and \( \theta(X_\alpha, t) \).

For \( i = 1, 2, 3 \), let \( \phi_i : \Omega \rightarrow \mathbb{R} \) be a smooth function such that \( \phi_i = 0 \) on \( \partial_1 \Omega \). Taking the inner product of both sides of Eq. (4.2a) with \( \phi_i \), integrating the resulting equation over \( \Omega \), using the divergence theorem on the first term on the right-hand side and boundary condition (4.6), we arrive at the following integral equation:

\[
\int_{\Omega} \rho_0 \dot{\phi}_i \, d\Omega = \int_{\partial_2 \Omega} f_i \phi_i \, d\Gamma - \int_{\Omega} T_{i\alpha} \phi_{i\alpha} \, d\Omega + \int_{\partial_1 \Omega} \rho_0 \partial_i \phi_i \, d\Omega
\]
Let $\eta : \overline{\Omega} \to \mathbb{R}$ be a smooth function such that $\eta = 0$ on $\partial_3 \Omega$. Multiplying both sides of Eq. (4.9) with $\eta$, integrating the resulting equation over $\Omega$, using the divergence theorem on the first term on the right-hand side and boundary condition (4.7), we get the following:

$$\int_{\Omega} \rho_0 \hat{\theta} \eta \, d\Omega = \int_{\partial_2 \Omega} h \eta \, d\Gamma + \int_{\Omega} Q_\alpha \eta_{,\alpha} \, d\Omega + \int_{\Omega} (T_{ia}^{ne} \hat{F}_{i\alpha} + \rho_0 r) \eta \, d\Omega$$  \hspace{1cm} (4.12)

We note that different quantities in Eqs (4.11) and (4.12) depend on only time histories of first-order derivatives with respect to $X_{,\alpha}$ of unknown functions $x_i(X_{,\alpha}, t)$ and $\theta(X_{,\alpha}, t)$.

Let

$$H^1 = \left\{ \phi | \phi : \overline{\Omega} \to \mathbb{R}^1 \bigg| \phi_{,\alpha} \phi_{,\alpha} < \infty \right\}$$ \hspace{1cm} (4.13)

$$T_m = \left\{ \phi_i | \phi_i \in H^1, \phi_i = 0 \text{ on } \partial_1 \Omega \right\}$$

$$S_m = \left\{ \phi_i | \phi_i : \overline{\Omega} \times (0, T) \to \mathbb{R}, \phi_i(\cdot, t) \in H^1, \phi_i(X_{,\alpha}, t) = \bar{x}_i(X_{,\alpha}, t) \text{ on } \partial_1 \Omega \right\}$$

$$T_\theta = \left\{ \phi | \phi \in H^1, \phi = 0 \text{ on } \partial_3 \Omega \right\}$$

$$S_\theta = \left\{ \phi | \phi : \overline{\Omega} \times (0, T) \to \mathbb{R}, \phi(\cdot, t) \in H^1, \phi(X_{,\alpha}, t) = \bar{\theta}(X_{,\alpha}, t) \text{ on } \partial_3 \Omega \right\}$$

$$S = S_m \otimes S_\theta, \quad T = T_m \otimes T_\theta$$  \hspace{1cm} (4.14)

Then, a weak formulation of the given problem can be stated as follows. Find $(x_i, \theta) \in S$ such that Eqs (4.11) and (4.12) hold for every $(\phi_i, \eta) \in S$. One can think of $(\phi_i, \eta) \in S$ as representing virtual displacements and virtual temperature fields. Functions in $S$ and $T$ have the same smoothness (differentiability) requirements and differ only in boundary conditions on $\partial_1 \Omega$ and $\partial_3 \Omega$. When $S \neq S$, the weak formulation is called Petrov–Galerkin. Let $g \in S$ be a fixed function. Then, every function $u \in S$ can be written as $u = g + v$ for some $v \in S$. Equivalently, $S = S \oplus \{g\}$. Regarding $g$ as a known function, the problem reduces to finding a $v \in S$ such that Eqs (4.11) and (4.12) hold for every $(\phi_i, \eta) \in S$, and the weak formulation is called Bubnov–Galerkin (or simply Galerkin in many books).

Let $S^n \subset S$ and $T^n \subset S$ be finite-dimensional sets. Then an approximate solution of the given problem is functions $(x^n_i, \theta^n) \in S^n$ such that Eqs (4.11) and (4.12) hold for every $(\phi^n_i, \eta^n) \in S^n$ with $T_{ia}, Q_\alpha$ and $T_{ia}^{ne}$ now evaluated from $(x^n_i, \theta^n)$ rather than from $(x_i, \theta)$.

Let $\psi_1, \psi_2, \ldots, \psi_n$ be basis functions in $T^n_\theta$ and $\zeta_1, \zeta_2, \ldots, \zeta_n$ be basis functions in $T^n_\theta$. Then,

$$\eta^n(X_{,\alpha}) = \psi_A(X_{,\alpha}) \eta_A, \quad A = 1, 2, \ldots, n$$  \hspace{1cm} (4.15a)

$$\theta^n(X_{,\alpha}, t) = \psi_A(X_{,\alpha}) \theta_A(t)$$  \hspace{1cm} (4.15b)

$$\phi^n_i(X_{,\alpha}) = \zeta_A(X_{,\alpha}) \phi_{Ai}, \quad i = 1, 2, 3$$  \hspace{1cm} (4.15c)

$$x^n_i(X_{,\alpha}, t) = \zeta_A(X_{,\alpha}) d_A(t)$$  \hspace{1cm} (4.15d)
Recall that a repeated index implies summation over the range of the index. Substitution from Eq. (4.15) into Eqs (4.11) and (4.12) and using the fact that these equations must hold for all choices of $\eta_A$ and $\phi_{A_i}$, we arrive at the following set of coupled ordinary differential equations (ODEs) in time:

\[
M_{AB}\ddot{d}_{Bi} = F^m_{Ai}
\]

(4.16)

\[
H_{AB}\dot{\theta}_B = F^\theta_A
\]

(4.17)

\[
F^m_{Ai} = \int_{\partial\Omega} f_i\zeta_A \, d\Gamma - \int_{\Omega} T_{i\alpha}\zeta_{A,\alpha} \, d\Omega + \int_{\Omega} \rho_0 b_i\zeta_A \, d\Omega
\]

(4.18a)

\[
F^\theta_A = \int_{\partial\Omega} h\psi_A \, d\Gamma + \int_{\Omega} Q_{\alpha}\psi_{A,\alpha} \, d\Omega + \int_{\Omega} G\psi_A \, d\Omega
\]

(4.18b)

\[
G = T^{\text{ne}}_{i\alpha}\dot{F}_{i\alpha} + \rho_0 r
\]

(4.18c)

\[
M_{AB} = \int_{\Omega} \rho_0\zeta_A\zeta_B \, d\Omega
\]

(4.18d)

\[
H_{AB} = \int_{\Omega} \rho_0 c\psi_A\psi_B \, d\Omega
\]

(4.18e)

In these equations, $T_{i\alpha}$, $Q_{\alpha}$ and $T^{\text{ne}}_{i\alpha}$ are functions of unknown fields $x_\alpha^n(X, t)$ and $\theta^n(X, t)$ or equivalently of $d_{A_i}(t)$ and $\theta_B(t)$.

From initial condition (4.8), we can similarly derive

\[
M_{AB}d_{Bi}(0) = F^m_{Ai}
\]

\[
M_{AB}\dot{d}_{Bi}(0) = \dot{F}^m_{Ai}
\]

\[
H_{AB}\theta_B(0) = \dot{F}^\theta_A
\]

(4.19)

and thus evaluate initial conditions for integrating, with respect to time $t$, ODEs (4.16) and (4.17).

Recall that the natural boundary conditions have been embedded in the weak formulation of the problem, and the essential boundary conditions are satisfied by either selecting a function $g$ or another equivalent method.

Because of using the referential description of motion, a weak formulation of Eq. (4.1) is not needed, and the mass matrix can be evaluated only once. The present mass density can be determined from the computed displacement field and Eq. (4.1).

Let $\Omega$ be divided into the union of disjoint subdomains $\Omega_e$ satisfying

\[
\Omega = \bigcup_{e=1}^m \Omega_e; \quad \Omega_e \cap \Omega_f = \phi, \quad e \neq f; \quad \overline{\Omega_e} \cap \overline{\Omega_f} \neq \phi
\]
where $\Omega_e$ and $\Omega_f$ are adjacent subdomains, and $\overline{\Omega}_e$ includes boundaries of $\Omega_e$. That is, adjacent subdomains share at most a common boundary. These subdomains are called finite elements (FEs), and they generally have straight boundaries (e.g., $\Omega_e$ is a straight line in one-dimensional (1D) space, is a triangle or a quadrilateral in a plane and is a cube or a tetrahedron in a 3D space). Points selected on the boundaries of FEs are called nodes. The collection of FEs and nodes is called an FE mesh or simply a mesh.

In the FE work, the number of basis functions $\{\psi_A\}$ (or $\{\zeta_A\}$) equals the number of nodes in the mesh. The basis functions are simple polynomials, and the basis function $\psi_A$ corresponding to node A equals 1 at node A, and 0 on all elements that do not meet at node A. Thus, $\psi_A$ equals zero at all nodes except node A and has a compact support. Equations (4.15c) and (4.15d) imply that $d_A(t)$ and $\theta_A(t)$ equal, respectively, $x(t)$ and $\dot{\theta}(t)$ at node A. The restrictions of basis functions to an FE are called shape functions. For the problem being studied here, the number of shape functions for an element equals the number of nodes on the element, and the shape function corresponding to node A of element $\Omega_e$ equals 1 at node A, 0 at all other nodes and vanishes on all sides of the element $\Omega_e$ that do not pass through node A. Generally, shape functions and basis functions are simple polynomials; the former are defined on an FE, and the latter are defined on the entire domain $\Omega$.

The integrals appearing in Eqs (4.18a) and (4.18b) are evaluated numerically. To illustrate the procedure, we consider one such integral:

$$I = \int_{\Omega} f \, d\Omega = \sum_e \int_{\Omega_e} f \, d\Omega \equiv \sum_e I^e$$

$$I^e = \int_{\Omega_e} f \, d\Omega$$

Let

$$T^e : \Omega_M \rightarrow \Omega_e; \quad \Omega_M = [-1, 1] \times [-1, 1] \times [-1, 1]$$

be a one-to-one, continuously differentiable and invertible map with continuously differentiable inverse. Here, we have tacitly assumed that $\Omega_e$ is a ‘brick’ or cubic element. We express $T^e$ as

$$X_\alpha = X_\alpha(\xi_f)$$

and require that

$$\det \left[ \frac{\partial X_\alpha}{\partial \xi_f} \right] > 0 \quad \text{for every} \quad \xi_f \in \Omega_M$$
With

\[ J = \det \left[ \frac{\partial X_\alpha}{\partial \xi_j} \right] \]  \hspace{1cm} (4.25)

the integral (4.21) can be written as:

\[ I^e = \int_{\Omega_e} f \, dX_1 \, dX_2 \, dX_3 = \int_{\Omega_M} f^J \, d\xi_1 \, d\xi_2 \, d\xi_3 = \int_{-1}^{1} d\xi_3 \int_{-1}^{1} d\xi_2 \int_{-1}^{1} f^J \, d\xi_1 \]  \hspace{1cm} (4.26)

Recall that for a polynomial of degree \(2n+1\),

\[ \int_{-1}^{1} f \, ds = \sum_{i=1}^{n+1} W_i f(s_i) \]  \hspace{1cm} (4.27)

Here, \(s_1, s_2, \ldots, s_{n+1}\) are sampling points or quadrature points and \(W_1, W_2, \ldots, W_{n+1}\) are the corresponding weights. The sampling points are roots of \(P_{n+1}(s) = 0\) and

\[ W_i = \int_{-1}^{1} L_i(s) \, ds \]  \hspace{1cm} (4.28)

where \(P_{n+1}\) is a Legendre polynomial of degree \(n+1\) and \(L_i\) is a Lagrange polynomial of degree \(n\) associated with point \(s_i\) and is defined by

\[ L_i(s) = \prod_{j=1}^{n+1}^j (s - s_j) \prod_{j=1}^{n+1}^j (s_i - s_j), \quad j \neq i \]  \hspace{1cm} (4.29)

Thus, when the integrand \(f^J\) in Eq. (4.26) is a polynomial, \(I^e\) can be evaluated exactly by using an appropriate quadrature rule in each coordinate direction. However, in general, the integrand is not a polynomial, and all integrals in Eqs (4.18a), (4.18b), (4.18d) and (4.18e) are evaluated approximately by employing a quadrature rule.
We note that the mapping $T^e$ in Eq. (4.22) is generally taken to be

$$X_\alpha = \sum_{A=1}^{8} N_A(\xi_1, \xi_2, \xi_3) X^A$$

(4.30a)

$$N_A(\xi_1, \xi_2, \xi_3) = \frac{1}{8} (1 + \xi^A_1 \xi_1)(1 + \xi^A_2 \xi_2)(1 + \xi^A_3 \xi_3)$$

(4.30b)

where $(\xi^A_1, \xi^A_2, \xi^A_3)$ are coordinates $(\pm 1, \pm 1, \pm 1)$ of node A of element $\Omega_M$ (a cube of side 2) in the local coordinate system, and $X^A_\alpha$ coordinates in the global coordinate system of the node of $\Omega_e$ that corresponds to node A of $\Omega_M$.

A field variable, for example, temperature $\theta$ defined on $\Omega_e$, can be expressed as a function of $\xi_1$, $\xi_2$ and $\xi_3$ through the transformation in Eq. (4.23). To evaluate constitutive quantities, we need gradients of $x_i(X_\alpha, t)$ and $\theta(X_\alpha, t)$ with respect to $X_\alpha$. To solve Eq. (4.30a) for $\xi_1$, $\xi_2$ and $\xi_3$, we note that

$$dX_\alpha = \frac{\partial N_A}{\partial \xi_i} d\xi_i X^A_\alpha$$

(summed on A and i)

$$= \frac{\partial N_A}{\partial \xi_i} X^A_\alpha \frac{\partial \xi_i}{\partial X_\beta} dX_\beta$$

Therefore,

$$\left[ \frac{\partial \xi_i}{\partial X_\beta} \right] = \left[ \frac{\partial N_A}{\partial \xi_i} X^A_\beta \right]^{-1}$$

Thus,

$$\frac{\partial \theta}{\partial X_\alpha} = \frac{\partial \theta}{\partial \xi_i} \frac{\partial \xi_i}{\partial X_\alpha}$$

(4.31)

can be computed.

### 4.2.1 Lumping of the Mass Matrix

The bandwidth of the consistent mass matrix computed from Eq. (4.18d) depends on the node-numbering scheme. To reduce computational effort, we generally approximate it by a lumped (or diagonal) matrix; the same is also done for the consistent heat capacity matrix given by Eq. (4.18e). The use of node points as
quadrature points to numerically evaluate Eq. (4.18d) will give a diagonal mass matrix but will result in some zero diagonal entries for axisymmetric problems. Zero or negative diagonal entries in mass matrix can have disastrous consequences. The following two techniques are commonly used to obtain a lumped mass matrix:

- **Row-sum technique**: Elements in each row of the consistent mass matrix are summed, and the result is placed on the diagonal. It can sometimes produce negative masses.
- **Special-lumping technique**: Entries of the lumped mass matrix are set proportional to corresponding diagonal elements of the consistent mass matrix with the constant of proportionality selected to conserve the total mass. The positive definiteness of the consistent mass matrix requires that its diagonal entries be positive.

### 4.2.2 Numerical Integration of ODEs

Equations (4.16) and (4.17) are non-linear, coupled ODEs in \( d_B \) and \( \theta_B \) because their right-hand sides are also functions of \( d_B \) and \( \theta_B \). In ASB problems, Jaumann’s rate of Cauchy’s stress tensor is usually expressed as a linear function of the elastic part of the strain-rate tensor. Invariably, one postulates evolution laws for internal variables describing microstructural changes such as porosity, dislocations and damage in the body. Timescales for these processes may differ by several orders of magnitude, which make the system of equations very stiff. For stiff ODEs, the ratio of the maximum eigenvalue to the minimum eigenvalue is very large.

For example [23],

\[
\dot{u} = 998u + 1998v \\
\dot{v} = -999u - 1999v
\]  

(4.32)

with initial conditions

\[
u(0) = 1, \quad v(0) = 0
\]  

(4.33)

have the solution

\[
u(t) = 2e^{-t} - e^{-1000t} \\
v(t) = -e^{-t} + e^{-1000t}
\]  

(4.34)

The term \( e^{-1000t} \) requires a time step of \( \leq 1/1000 \) for the method to be stable. However, this term makes negligible contributions to values of \( u \) and \( v \) for \( t > 0 \). The generic problem with stiff equations is that for the forward-difference integration scheme to be stable, we need to follow the variation in the solution on the
shortest timescale. We can overcome this by evaluating the time derivative with a backward-difference scheme. For example, consider

\[ \dot{y} = -cy \] (4.35)

Then,

\[ y_{n+1} = \begin{cases} 
    \frac{y_n}{1 + c\Delta t}, & \text{backward-difference} \\
    y_n(1 - c\Delta t), & \text{forward-difference}
\end{cases} \] (4.36)

where \( y_n \approx y(t_n) \). Clearly the forward-difference method is unstable if \( \Delta t > 2/c \), but the backward-difference method is unconditionally stable (i.e., stable for all values of \( \Delta t \)). We sacrifice accuracy by using large time steps, but we maintain stability.

For a system of linear equations with constant coefficients,

\[ \dot{y} = -Cy \] (4.37)

where \( C \) is a positive definite matrix; the explicit method is stable only if

\[ \Delta t < \frac{2}{\lambda_{\text{max}}} \] (4.38)

where \( \lambda_{\text{max}} \) is the largest eigenvalue of \( C \). The backward-difference scheme gives

\[ y_{n+1} = (I + C\Delta t)^{-1}y_n \] (4.39)

and is stable for all values of \( \Delta t \). For a system of non-linear equations,

\[ \dot{y} = f(t, y) \] (4.40)

The backward differencing gives

\[ y_{n+1} = y_n + \Delta t f(t_{n+1}, y_{n+1}) \]

\[ \approx y_n + \Delta t \left[ f(t_{n+1}, y_n) + \frac{\partial f}{\partial y} \bigg|_{y_n} \cdot (y_{n+1} - y_n) \right] \] (4.41)
and requires the inversion of \(1 - \Delta t \frac{\partial f}{\partial y}\), which is computationally expensive. By averaging the explicit and the implicit first-order methods, we get a second-order accurate method:

\[
y_{n+1} = y_n + \frac{\Delta t}{2} \left[ f(t_{n+1}, y_n) + \frac{\partial f}{\partial y} \bigg|_{y_{n+1}} \cdot (y_{n+1} - y_n) + f(t_n, y_n) \right]
\]  

(4.42)

One can use either LSODE (Livermore Solver for ODEs) or VODEPK (Variable Coefficient ODE Solver package) to numerically integrate a stiff set of ODEs. These solvers adaptively adjust the time-step size to compute a solution of the ODEs within prescribed absolute and relative tolerances. In strain localization problems, the time-step size drops dramatically after the deformation begins to localize, indicating an increase in the stiffness of the ODEs.

### 4.2.3 Satisfaction of Essential Boundary Conditions

Instead of explicitly using the function \(g\), Eqs (4.16) and (4.17) are assembled for all nodes and are then modified as follows to satisfy the essential boundary conditions prescribed at node A. From the given time history of the displacement at node A, one computes its acceleration at any time \(t\). For a lumped mass matrix, the displacement boundary condition at node A is satisfied when \(F_{Ai}\) is replaced by \(M_{AA} \ddot{d}_{Ai}\) (no sum on \(A\)). One follows this procedure for every node where an essential boundary condition is given. For a consistent mass matrix, one can use the penalty method in which \(M_{AA}\) is replaced by \(M_{AA} + \lambda\) and \(F_{Ai}\) by \(\lambda \ddot{d}_{Ai}\), where \(\lambda\) is 10^5 times the largest entry in the mass matrix. The accuracy in the satisfaction of the essential boundary condition can be improved by increasing the value of \(\lambda\).

### 4.2.4 Interpretation of the FE Solution

The computed nodal values of displacements and temperatures are reasonably accurate. However, stresses, heat flux and other quantities involving derivatives of displacements and temperatures are evaluated at quadrature points, which are generally in the interior of \(\Omega_e\). Then, they can be extrapolated or interpolated to other points of interest, for example, see Refs [24–26].

### 4.2.5 Factors Affecting Quality of the Approximate Solution

The following list of factors that affect the quality of the solution is not exhaustive:

1. **Dimensionality of the finite-dimensional space**: The quality of the approximate solution generally improves with an increase in the dimensionality (number of nodes) of the finite-dimensional space (recall the convergence to the function of its Fourier series with an increase in the number of terms in the series).
2. **Choice of basis functions**: For localization problems, lowest order basis functions are recommended.

3. **FE mesh design**: Adaptively refined meshes with element size inversely proportional to a scalar measure of deformation within an element are recommended. Care should be taken to properly grade the mesh to avoid a very large element abutting an extremely small one.

4. **Number of quadrature points used to evaluate integrals**: The CPU time increases with an increase in the number of quadrature points. However, too few quadrature points may result in hourglass or spurious modes.

5. **Consistent/lumped mass matrix**: Consistent mass matrices generally require a smaller time-step size for stability than lumped mass matrices.

6. **Explicit/implicit time integration scheme**: Numerical integration schemes generally introduce time-period errors and dissipation. For the 1D linear elastic problem, a uniform FE mesh, a lumped mass matrix and the explicit (central-difference) integration scheme with $\Delta t = h/c$ give exact values of nodal displacements. Here, $h$ is the element size and $c$ is the speed of the wave in the bar.

7. **Tolerances**: Tolerances in solving non-linear algebraic equations if an implicit time integration scheme is used.

8. **Time-step size**: As $\Delta t \to 0$, an approximate solution of Eqs (4.16) and (4.17) converges to the analytical solution of the discretized problem and not of the continuous problem defined by Eqs (4.1)–(4.3).

9. **Points where stresses/fluxes are computed for post-processing of results**: It is recommended that fluxes be computed at quadrature points, at the element centroid or at the Barlow points [24–26].

10. **Time integration**: The time integration scheme should maintain objectivity of the rate form of constitutive relations.

### 4.3 Analysis of 1D Shear Band Problems

Marchand and Duffy [27] analysed the initiation and growth of ASBs by twisting thin-walled tubes that had varying thickness because of machining errors. As a first approximation, torsional deformations of the tube can be regarded as equivalent to simple shearing deformations of a representative volume element situated near the tube’s midsection. The material presented in this section has been taken from Refs [28–30]; other investigations are cited in these articles. The goals here are to define ASB initiation criterion and bandwidth, and characterize effects of different viscoplastic constitutive equations, thermal conductivity, inertia forces and defect size.

We choose a fixed set of rectangular Cartesian coordinate axes with origin at the lower surface of the block and the direction of shearing along the $x$-axis, for example, see Figure 4.1. We assume that particles move only in the $x$-direction, and a particle’s $x$-velocity, $v$, and temperature rise, $\theta$, are functions of $y$ only. These are found by solving

$$\rho w(y)\dot{v} = (w(y)s)_y, \quad 0 < y < H$$

(4.43)
Here, \( s \), \( \gamma_p \), \( w \), \( k \) and \( \mu \) represent, respectively, the shear stress, plastic strain, the block thickness, the thermal conductivity and the shear modulus. We have used Fourier’s law of heat conduction and assumed an additive decomposition of the shear strain rate into elastic and plastic parts. The evolution of the shear stress is governed by the rate form of Hooke’s law; all of the plastic working is converted into heating and have neglected effects, if any, of thermal stresses. The last assumption is reasonable because deformations are driven by the shear stress. We note that simple shear deformations are isochoric, i.e., volume preserving. Thus, the mass density does not change during the deformation process. We solve the problem for four different viscoplastic relations in Eq. (4.46) (i.e., function \( g \)).

We non-dimensionalize variables as follows:

\[
\begin{align*}
\bar{y} &= y/H, \quad \bar{t} = \nu_0 t/H, \quad \bar{w} = w/H, \quad \bar{\theta} = \theta/\theta_0, \quad \theta_0 = \sigma_0/\rho c_R \\
\bar{s} &= s/\sigma_0, \quad \bar{\mu} = \rho \nu_0^2/\sigma_0, \quad \bar{k} = k/(\rho \nu_0 c_R H) \\
\bar{c} &= c(\theta)/c_R, \quad \bar{\mu} = \mu/\sigma_0, \quad c_R = c(\theta_R), \quad \theta_R = 300 \text{ K}
\end{align*}
\] (4.47)

Here, \( \nu_0 \) is the steady value of the shearing velocity applied to the top surface of the block when its lower surface is kept fixed. Furthermore, \( \bar{t} \) equals the average strain induced in the specimen at time \( t \), \( \theta_0 \) is the temperature rise at a point due to
plastic strain of 1 when deformations are locally adiabatic and the flow stress is kept constant at \( \sigma_0 \), and \( \rho \) is the ratio of the kinetic energy density to the plastic work done per unit volume at constant flow stress \( \sigma_0 \) and plastic strain of 2. In terms of non-dimensional variables, Eqs (4.43)–(4.46) remain unchanged except that dimensional quantities are replaced by non-dimensional ones, and the problem domain becomes \( 0 < \bar{y} < 1 \). Henceforth, in this section, only non-dimensional variables are used, and superimposed bars have been dropped.

For initial and boundary conditions, we take the following:

\[
\begin{align*}
\theta(y,0) &= 0, \quad v(y,0) = 0, \quad s(y,0) = 0, \quad \gamma_p(y,0) = 0 \\
\theta_y(0,t) &= 0, \quad \theta_y(1,t) = 0, \quad v(0,t) = 0 \\
v(1,t) &= t/0.01, \quad 0 \leq t \leq 0.01 \\
\theta(y,1) &= 1, \quad t \geq 0.01
\end{align*}
\] (4.48)

That is, the block is initially stress free, undeformed, at rest and has a uniform temperature, normalized to be zero. The overall deformations of the block are taken to be adiabatic, and the lower surface is at rest, whereas the upper surface is assigned a velocity that increases from 0 to 1 in a non-dimensional time of 0.01 and then stays at 1.0. We note that the prescribed \( x \)-velocity at surfaces \( y = 0 \) and \( y = 1 \) is neither symmetric nor antisymmetric about the midsurface, \( y = 1/2 \). The block is taken to be thinnest at \( y = 1/2 \) and thickest at \( y = 0, 1 \), with the thickness variation given by

\[
w(y) = w_0 \left[ 1 + \frac{\delta}{2} \sin \left( \frac{1}{2} + 2y \right) \pi \right] \] (4.49)

Marchand and Duffy [27] reported nearly 10% variation in the thickness of steel tubes they tested in torsion. Our choice of locating the thinnest section at \( y = 1/2 \) is for convenience only and should not affect results.

### 4.3.1 Viscoplastic Flow Rules

**Wright–Batra Relation**

Wright and Batra [31] modified the Litonski relation [32] to account for elastic unloading of a material point and postulated the following:

\[
\dot{\gamma}_p = \Lambda s
\] (4.50)

\[
\Lambda = \max \left[ 0, \left( \left( \frac{s}{(1 - v\theta)(1 + \psi/\psi_0)^n} \right)^{1/m} - 1 \right) / bs \right]
\] (4.51)

\[
\dot{\psi} = s\dot{\gamma}_p / (1 + \psi/\psi_0)^n
\] (4.52)
Here, $\psi$ may be thought of as an internal variable that describes work hardening of the material. Equation (4.52) implies that the rate of growth of $\psi$ is proportional to plastic working. In Eq. (4.51), $(1 - \nu \theta)$ describes the material softening due to its being heated up, $b$ and $m$ characterize its strain-rate sensitivity and $\psi_0$ and $n$ describe its work hardening. Equations (4.50) and (4.51) imply the following:

$$\dot{\gamma}_p = 0 \quad \text{if} \quad s \leq (1 - \nu \theta)(1 + \psi/\psi_0)^n$$  \hspace{1cm} (4.53)

Thus, if the local state given by $(s, \psi, \theta)$ lies inside or on the loading surface, $s = (1 - \nu \theta)(1 + \psi/\psi_0)^n$, then the plastic strain rate vanishes, and the material deforms elastically. However, if the point $(s, \psi, \theta)$ is outside the loading surface, then the plastic strain rate is adjusted so that the local state lies on the yield surface, $s = (1 - \nu \theta)(1 + \psi/\psi_0)^n(1 + b\dot{\gamma}_p)$. Besides $\sigma_0$, which has been used to non-dimensionalize quantities, five parameters, $\nu$, $b$, $m$, $\psi_0$ and $n$, are needed to specify the viscoplastic response of the material.

**Bodner–Partom Relation**

Bodner and Partom [33] assumed that there is no loading surface and plastic strain rate, $\dot{\gamma}_p$, albeit very small at low values of $s$, is always positive. Their constitutive relation can be written as:

$$\dot{\gamma}_p = D_0 \exp \left[ -\frac{1}{2} \left( \frac{z^2}{3s^2} \right)^n \right], \quad n = \frac{a}{T} + b$$  \hspace{1cm} (4.54)

$$z = z_1 - (z_1 - z_0) \exp(-mW_p)$$  \hspace{1cm} (4.55)

$$\dot{W}_p = s \dot{\gamma}_p$$  \hspace{1cm} (4.56)

Here, $T$ is the absolute temperature of a material particle, $W_p$ is the plastic work done, $z$ is an internal variable and $D_0$ is the limiting value of $\dot{\gamma}_p$, usually taken as $10^8$ s$^{-1}$. Besides $D_0$, we need values of $a$, $z_1$, $z_0$, $m$ and $b$ to characterize the material.

**Johnson–Cook Relation**

Johnson and Cook [34] tested 12 materials in simple shear and compression at different strain rates and found that

$$r = \left[ \frac{s}{(A + B\dot{\gamma}_p^n)(1 - T^n)} - 1.0 \right]$$

$$\dot{\gamma}_p = \dot{\gamma}_0 \exp \left( \frac{r}{C} \right), \quad r \geq 0$$

$$= 0, \quad r < 0$$  \hspace{1cm} (4.57)
\[ T = \frac{(\theta - \theta_0)}{(\theta_m - \theta_0)} \]  

(4.58)

describe well the test data. For \( \theta_0 \) equal to the ambient temperature and \( \gamma_0 = 1/\text{sec} \), they tabulated values of \( A, B, n, m, \theta_m \) and \( C \) for 12 materials. One can regard \( r = 0 \) as the loading surface.

**Power Law**

Klopp et al. [35] and Marchand and Duffy [27] have described the stress–strain curve for simple shear loading by:

\[ s = s_0 \left( \frac{\gamma}{\gamma_0} \right)^n \left( \frac{\dot{\gamma}_p}{\dot{\gamma}_0} \right)^m \left( \frac{\theta}{\theta_0} \right)^\nu \]  

(4.59)

where \( \gamma_y \) is the strain at yield in a quasi-static simple shear test at \( \dot{\gamma}_0 = 10^{-4} \text{ s}^{-1} \), parameters \( n \) and \( m \) characterize the strain and strain-rate hardening of the material and \( \nu < 0 \) is its thermal softening.

### 4.3.2 Determination of Values of Material Parameters

Batra and Kim [28] assumed values of material parameters to be temperature independent, ignored effects of phase transformations and determined their values by solving an initial-boundary-value problem (IBVP) that closely simulated test conditions of Marchand and Duffy [27] by ensuring that the computed shear stress/shear strain curve matched well the test data. Values of material parameters so determined are not unique because the underlying IBVP may not have a unique solution. Values of the quasi-static yield stress and the strain hardening exponent were kept close to those reported by Marchand and Duffy [27]. For HY100 steel, \( \rho = 7860 \text{ kg/m}^3 \), \( \sigma_0 = 405 \text{ MPa} \), \( k = 49.73 \text{ W/(m}^2 \text{ °C}) \) and \( c = 473 \text{ J/(kg °C)} \); values of other material parameters are listed below:

- **Wright–Batra relation**: \( \nu = 0.00185 \text{ °C}^{-1} \), \( \psi_0 = 4.86 \text{ MPa} \), \( m = 0.0117 \), \( n = 0.107 \), \( b = 10^4 \text{ s} \).
- **Bodner–Partom relation**: \( D_0 = 10^8 \text{ s}^{-1} \), \( z_1 = 3.95\sigma_0 \), \( z_2 = 3.21\sigma_0 \), \( m = 5/\sigma_0 \), \( a = 1200 \text{ K} \) and \( b = 0 \).
- **Johnson–Cook relation**: \( \dot{\gamma}_0 = 3300 \text{ s}^{-1} \), \( A = 0.45 \), \( B = 1.433 \), \( C = 0.0227 \), \( n = 0.107 \), \( m = 0.7 \), \( \theta_m = 1500 \text{ K} \) and \( \theta_0 = 300 \text{ K} \).
- **Power law**: \( \dot{\gamma}_0 = 10^{-4} \text{ s}^{-1} \), \( m = 0.0117 \), \( n = 0.107 \), \( \nu = -0.75 \), \( \theta_0 = 300 \text{ K} \), \( \gamma_y = 0.012 \).

For an average strain rate of 3300 \text{ s}^{-1} \), **Figure 4.2** shows the computed shear stress versus shear strain curves with the above given values of material parameters. Because of the magnified vertical scale, differences among these curves are exaggerated. Values of the maximum shear stress and the average strain at which they occur equal \((1.60, 0.25)\), \((1.55, 0.27)\), \((1.53, 0.27)\) and \((1.50, 0.26)\),
respectively, for the Wright–Batra, the power law, the Bodner–Partom and the Johnson–Cook relations.

4.3.3 Results

Computational Considerations

Governing Eqs (4.43) and (4.44) with function $g$ given by one of the flow rules are highly non-linear and difficult to solve analytically under side conditions given as Eq. (4.48); their approximate solution has been computed numerically by using the FEM as described in Section 4.2. The software LSODE is used to integrate with respect to time $t$ the resulting non-linear coupled ODEs. The block thickness at the centre is taken to be 90% of that at the edges.

ASB Initiation Criterion

Batra and Kim [36] proposed the following ASB initiation criterion. An ASB initiates at a material point when the shear stress there has dropped to 90% of its maximum value at that point; the material point is deforming plastically and deformations in its neighbourhood are inhomogeneous. We recall that in Marchand and Duffy’s [27] experimental work, the torque dropped when an ASB initiated. Figure 4.3A and B shows the dependence on the nominal strain rate, of the average shear strain, $\gamma_I$, corresponding to the maximum shear stress, and to the average shear strain, $\gamma_{0.9}$, when an ASB initiates. For the Bodner–Partom relation, $\gamma_I$ and $\gamma_{0.9}$ monotonically increase with an increase in the nominal strain rate, $\dot{\gamma}_{\text{avg}}$. For the other three viscoplastic relations, the dependence of $\gamma_I$ and $\gamma_{0.9}$ on $\dot{\gamma}_{\text{avg}}$ is not monotonic.

Figure 4.2 Shear stress versus shear strain curves for homogeneous simple shearing deformations of HY100 steel deformed at a strain rate of 3300 s$^{-1}$. Source: From Ref. [30].
In their torsional tests, Marchand and Duffy [27] called the width, $\delta_{\text{ASB}}$, of the intensely deformed region of uniform plastic strain as the ASB width. Batra and Chen [30] equated $\delta_{\text{ASB}}$ to the width of the region over which the plastic strain varied by no more than 10% of its maximum value in that region. In Figure 4.4, we have plotted $\delta_{\text{ASB}}$ as a function of $s/s_{\text{max}}$ for the four viscoplastic relations and $\dot{\gamma}_{\text{avg}} = 1000 \text{ s}^{-1}$. Whereas for the Wright–Batra and the Johnson–Cook relations, $\delta_{\text{ASB}}$ slowly decreases with the drop of the shear stress at the band centre; for the power law, it increases. For the Bodner–Partom relation, $\delta_{\text{ASB}}$ increases slightly when the shear stress drops from $0.7s_{\text{max}}$ to $0.5s_{\text{max}}$. Thus, even though the ASB

**Figure 4.3** For the four viscoplastic relations, dependence on the average shear strain rate of the shear strain corresponding to the maximum shear stress and to that when the shear stress has dropped to 90% of its maximum value.  
*Source:* From Ref. [30].

**Figure 4.4** For the four viscoplastic relations, evolution of the shear band width while the shear stress is dropping.  
*Source:* From Ref. [30].

**ASB Width**

In their torsional tests, Marchand and Duffy [27] called the width, $\delta_{\text{ASB}}$, of the intensely deformed region of uniform plastic strain as the ASB width. Batra and Chen [30] equated $\delta_{\text{ASB}}$ to the width of the region over which the plastic strain varied by no more than 10% of its maximum value in that region. In Figure 4.4, we have plotted $\delta_{\text{ASB}}$ as a function of $s/s_{\text{max}}$ for the four viscoplastic relations and $\dot{\gamma}_{\text{avg}} = 1000 \text{ s}^{-1}$. Whereas for the Wright–Batra and the Johnson–Cook relations, $\delta_{\text{ASB}}$ slowly decreases with the drop of the shear stress at the band centre; for the power law, it increases. For the Bodner–Partom relation, $\delta_{\text{ASB}}$ increases slightly when the shear stress drops from $0.7s_{\text{max}}$ to $0.5s_{\text{max}}$. Thus, even though the ASB
initiates when \( s/s_{\text{max}} = 0.9 \), its full development depends on the viscoplastic relation employed. Dilellio and Olmstead [37] used asymptotic methods to analyse boundary layers near edges of a specimen deformed in simple shear and modelled the material response by a power law type strain hardening and an exponential thermal-softening relation; their computed time history of evolution of \( \delta_{\text{ASB}} \) qualitatively agrees with that in Figure 4.4 for the power law.

For the four constitutive relations, Figure 4.5A and B exhibits the dependence of \( \delta_{\text{ASB}} \) on \( \dot{\gamma}_{\text{avg}} \) for \( s/s_{\text{max}} = 0.9 \) and 0.6. In each case, \( \delta_{\text{ASB}} \) first decreases with an increase in \( \dot{\gamma}_{\text{avg}} \) from \( 10^2 \) s\(^{-1}\) to about \( 10^{2.8} \) s\(^{-1}\) and then increases. For \( s/s_{\text{max}} = 0.9 \) and \( \dot{\gamma}_{\text{avg}} = 10^2 \) s\(^{-1}\), \( \delta_{\text{ASB}} \) computed with the Wright–Batra relation is the minimum, \( \delta_{\text{min}} \), and that with the Johnson–Cook relation, the maximum, \( \delta_{\text{max}} \), and \( \delta_{\text{max}}/\delta_{\text{min}} = 7 \). For \( s/s_{\text{max}} = 0.66 \) and \( \dot{\gamma}_{\text{avg}} = 1600 \) s\(^{-1}\), Marchand and Duffy [27] found that \( \delta_{\text{ASB}} \) varied between 20 and 55 \( \mu \)m around the circumference of the tube; Batra and Chen’s [30] computations give \( \delta_{\text{ASB}} = 2.3, 5.5, 14 \) and 6 \( \mu \)m for the Wright–Batra, the Johnson–Cook, the power law and the Bodner–Partom relations.

**Effect of Thermal Conductivity**

For \( \dot{\gamma}_{\text{avg}} = 3300 \) s\(^{-1}\), the Johnson–Cook relation and thermal conductivity, \( k = 0, 5, 50, 500 \) and 5000 W/(m \(^{\circ}\)C), Figure 4.6 displays the average shear stress, \( s_{\text{avg}} \), and the homologous temperature, \( \bar{T} \), versus the average shear strain, \( \gamma_{\text{avg}} \), in the specimen; results for the other two constitutive relations are qualitatively similar to those shown in Figure 4.6. The value of \( \gamma_{\text{avg}} \) at which the \( s_{\text{avg}} \) begins to drop increases a little with an increase in the value of \( k \). However, the rate of stress drop decreases dramatically as the value of \( k \) is increased from 50 to 500 W/(m \(^{\circ}\)C) as compared to that when \( k \) is increased from 5 to 50 W/(m \(^{\circ}\)C). The rate of temperature rise is steepest for \( k = 0 \) and decreases with an increase in the value of \( k \). As the value of
For five values of thermal conductivity and the Johnson–Cook relation, evolution with the average strain of the average shear stress (A) and the homologous temperature (B). 

Source: From Ref. [38].

$k$ increases, the heat conducted away from the central, hotter region to the colder, outer parts of the specimen increases, and the rate of temperature rise at the specimen centre decreases. Because of global adiabatic conditions assumed, the temperature everywhere in the specimen increases. Thus, the pre-localization response of a material is essentially the same whether or not heat conduction is considered, but the post-localization response is noticeably influenced by the consideration of thermal conductivity.

**Remarks**

Results presented in subsection 4.3.3 suggest that, for most materials, except possibly for copper whose thermal conductivity is high, it is reasonable to assume that the deformations are locally adiabatic prior to the initiation of a shear band. However, the post-localization deformations are strongly influenced by thermal conductivity. Batra [39,40] has shown through numerical experiments that the consideration of inertia forces delays the ASB initiation. Batra and Kim [29] found that for large values of the thermal-softening coefficient, $\nu$, the sudden collapse of the shear stress within the band results in an unloading elastic wave propagating outwards from the shear-banded region. Batra and Kim [32] have analysed the effect of the defect size on the initiation and growth of shear bands in 12 materials. They found that the bandwidth, computed when the shear stress has dropped to 95% of its peak value, does not correlate well with the thermal conductivity of the material. The band begins to grow rapidly when the shear stress has dropped to 90% of its peak value. Larger defects result in more severe localization of deformation for the same percentage drop in the shear stress. However, the defect size influences very little the homologous temperature when the shear stress either has attained its maximum value or has dropped to 85% of its maximum value.
Kim and Batra [41] have accounted for the dependence of material properties upon the temperature, and they found that the bandwidth and the average strain at which an ASB forms decrease with a decrease in the initial temperature of the specimen.

### 4.4 Adaptive Mesh Refinement for 2D Problems

Because of intense deformations of the material in an ASB, the FE mesh is severely distorted, and an interior angle of an FE can become either too small or too large. Thus, the deforming region should be frequently remeshed to capture late stages of ASB development. One way to accomplish this is to adaptively refine the mesh. Here we discuss two such techniques for 2D problems; the \( h \)-method in which the element size is varied \([42]\) and the \( r \)-method wherein the mesh topology (number of elements and nodes connecting various elements) is kept fixed, but locations of nodes are varied \([43]\).

#### 4.4.1 The \( h \)-Method

A coarse mesh is used to numerically solve the problem. This mesh is refined so that

\[
a_e = \int_{\Omega_e} I \, d\Omega, \quad e = 1, 2, \ldots, n_{el}
\]

is nearly the same for each FE \( \Omega_e \). In Eq. (4.60), \( I \) is the second invariant of the deviatoric strain-rate tensor and \( n_{el} \) equals the number of elements in the coarse mesh. Because points where the solution exhibits sharp gradients are unknown \textit{a priori}, the starting coarse mesh may be chosen as uniform. The motivation for making \( a_e \) the same over each element \( \Omega_e \) is that within the region of localization of deformation, values of \( I \) are very high as compared to those in the remaining region. Other variables such as the temperature rise, the maximum principal strain and the equivalent strain, which are also quite large within the band, will be suitable replacements for \( I \) in Eq. (4.60). The topology of the refined mesh will depend on the variable used in Eq. (4.60). To refine the mesh, we find the following:

\[
\bar{a} = \frac{1}{n_{el}} \sum_{e=1}^{n_{el}} a_e, \quad \bar{\varepsilon}_e = \frac{a_e}{\bar{a}}, \quad h_e = \frac{\bar{h}_e}{\bar{\varepsilon}_e}, \quad \text{and} \quad H_n = \frac{1}{N_e} \sum_{e=1}^{N_e} h_e, \quad n = 1, 2, \ldots, n_{ed}
\]

(4.61)

Here, \( \bar{h}_e \) is the size of element \( \Omega_e \) in the coarse mesh, \( N_e \) equals the number of elements meeting at node \( n \) and \( n_{ed} \) equals the number of nodes in the coarse mesh. We refer to \( H_n \) as the nodal element size at node \( n \).
To generate a new mesh, we first discretize the boundary by following the procedure proposed by Cescotto and Zhou [44]. Let AB be a segment of the contour to be discretized, $s$ the arc length measured from point A, and $H_A$ and $H_B$ the nodal element sizes for nodes located at points A and B, respectively. Knowing the values of $H$ at nodes in the coarse mesh, we define on AB a piecewise linear continuous function $H(s)$ that takes values at nodes given in Eq. (4.61). To discretize AB for the new mesh, we start from point A if $H_A < H_B$; otherwise, we start from point B. Let A be the starting point. We first find temporary positions of nodes on segment AB by using the following recursive procedure. Assume that points 1, 2, ..., $k$ have been found. Then the temporary location of point $(k+1)$ is given by:

$$s_{k+1} = s_k + \frac{1}{2} \left[H(s_k) + H(s_1^*)\right]$$  \hspace{1cm} (4.62)

where

$$s_1^* = s_k + H(s_k)$$  \hspace{1cm} (4.63)

Referring to Figure 4.7, the preceding procedure will result in the following four alternatives: $a = b = 0$, $a > b$, $b = a$, $a = b \neq 0$. If $a = b = 0$, then the temporary locations of nodes are their final positions. Depending on whether $a < b$ or $b \leq a$, nodes 2 to $p$ or 2 to $p+1$ are moved, the displacement of a node being proportional to the value of $H$ there, so that either node $p$ or node $(p+1)$ coincides with point B. This determines final positions of nodes on segment AB.

Having discretized the boundary, we use the concept of an advancing front [45–47] to generate elements. An advancing front consists of straight line segments that are available to form a side of an element. Thus, to start with, it consists of the discretized boundary. We choose the smallest line segment (say side AB in

![Figure 4.7](image.png)

Figure 4.7 (A) Discretization of a boundary segment for mesh refinement and (B) advancing front and new element generation.
Figure 4.7) connecting two adjoining nodes and determine the nodal element size

\[ H_M = \frac{H_A + H_B}{2} \]

at the midpoint M of AB. We set

\[
\delta = \begin{cases} 
0.8\overline{AB} & \text{if } H_M < 0.8\overline{AB} \\
H_M & \text{if } 0.8\overline{AB} \leq H_M \leq 1.4\overline{AB} \\
1.4\overline{AB} & \text{if } 1.4\overline{AB} < H_M 
\end{cases}
\]  

(4.64)

and find point \( C_1 \) at a distance \( \delta \) from points A and B (cf. Figure 4.7). Here \( \overline{AB} \) equals the length of segment AB, and numbers 0.8 and 1.4 can be changed to generate elements of different sizes. We search for all nodes on the active front that lie inside the circle with centre at \( C_1 \) and radius \( \delta \). At the end of this list are added points \( C_1, C_2, C_3, C_4 \) and \( C_5 \), which lie on \( C_1M \) and divide it into five equal parts. We next determine the first point \( C \) in the list that satisfies the following three conditions:

1. Area of triangle ABC is \( > 0 \).
2. Sides AC and BC do not cut any of the existing sides in the front.
3. If any of the points \( C_1, C_2, \ldots, C_5 \) are chosen, that point is not too close to the front.

The triangle ABC is an element in the new mesh. If \( C \) is one of the points \( C_1, C_2, \ldots, C_5 \), then a new node is also created. The advancing front is updated by removing the line segment AB from it and adding line segments AC and CB to it. The element-generation process ceases when there is no side left in the active front.

We determine values of solution variables at a newly created node by first finding out to which element in the coarse mesh this node belongs and then interpolating values of solution variables at this node. This process and that of searching for line segments and points in the previous element-generation technique require considerable CPU time. These operations are optimized to some extent by using the heap list algorithm (e.g., see Ref. [48]) for deleting and inserting new line segments, quad tree structures, linked lists for searching line segments and points and also for interpolation of solution variables at newly created nodes.

### 4.4.2 The r-Method

The goal here is to reposition nodes so that \( a_e \) defined by Eq. (4.60) is nearly the same for each element \( \Omega_e \). Having solved the problem with an initial mesh it is refined as follows. We begin with either the horizontal or the vertical boundary and relocate nodes on it; we describe the process starting with the left vertical edge. After having repositioned nodes on it, we do the same on the almost vertical curve that passes through nodes next to the left vertical side, and continue the process till we reach the right vertical edge. The procedure is then repeated beginning with the top or the bottom horizontal edge and going to the other end.

Referring to Figure 4.8, let AB be the curve on which nodes are to be relocated. We plot the distribution of \( I \) on AB with abscissa as the distance of a point from
A measured along AB and ordinate as the value of $I$ at that point. Values of $I$ at numerous points on AB are obtained by linear interpolation from values at node points. If $S$ equals the total area under the curve, then approximate location, $s_n^a$, of the $n^{th}$ node on AB is given by

$$
\int_{s_{n-1}}^{s_n^a} I \, ds = \frac{S}{N_{cs}}
$$

Figure 4.8 (a) Curve AB on which nodes are to be relocated; (b) distribution of $I$ on curve AB; (c) temporary position on curve AB of relocated nodes; (d) reposition nodes on curve AB; (e) relocation of an interior node to smoothen out the generated mesh.
where \( N_{ex} \) equals the number of elements on AB. We reposition the node to the interpolation point immediately to the left of its approximate location determined by Eq. (4.65). In Figure 4.8c, the position of a node as found by Eq. (4.65) is shown by a superimposed prime, and its relocation is shown in Figure 4.8d by superimposed two primes. Because end points on AB are kept fixed, the previously given procedure can be employed by starting from either node A or node B. Note that when nodes on an approximate horizontal curve are relocated, positions of nodes A and B will change.

Quadrilateral elements produced by the preceding technique are not always well shaped in the sense that an interior angle may be either too small or too large. It usually happens in regions where the element size varies noticeably. We use the mesh smoothing method of Zhu et al. [49] to improve upon shapes of quadrilateral elements. Each interior node is repositioned to the centroid of the polygon formed by all elements meeting at the node. As illustrated in Figure 4.8e, the internal node \( i \) is moved to \( i' \) with coordinates given by:

\[
x_i' = \frac{1}{4M} \sum_{a=1}^{M} (x_j + 2x_k + x_l) \alpha_a, \quad y_i' = \frac{1}{4M} \sum_{a=1}^{M} (y_j + 2y_k + y_l) \alpha_a
\]

(4.66)

where \( M \) is the number of elements sharing node \( i \). After having relocated all internal nodes, element shapes are checked to see if all interior angles of every element are between \( 20^\circ \) and \( 160^\circ \); these limiting values of interior angles are arbitrarily chosen. If not, nodes are repositioned according to Eq. (4.66) till such is the case. Because of smoothening of the mesh, the value of \( a_e \) defined by Eq. (4.60) is only approximately the same for all elements in the mesh.

### 4.4.3 Numerical Results

The two preceding adaptive mesh-refinement techniques have been used to delineate the initiation and development of ASBs in plane-strain compression of a square block. Because of the presumed symmetry of deformations about the horizontal and the vertical centroidal axes, deformations of only a quarter of the block are investigated. The constitutive relation for the material is taken to be

\[
\sigma_{ij} = -B \left( \frac{\rho}{\rho_0} - 1 \right) \delta_{ij} + \frac{\sigma_0}{\sqrt{3I}} (1 + bI)^{n(1 - v\theta)} D_{ij}
\]

(4.67)

\[
2I^2 = D_{ij} D_{ij}, \quad D_{ij} = D_{ij} - \frac{1}{3} D_{kk} \delta_{ij}
\]

(4.68)

Here, \( \sigma_{ij} \) is the Cauchy stress tensor, \( D \) is the strain-rate tensor and \( B \) may be thought of as the bulk modulus for the material of the block. Batra [50] proposed Eq. (4.67)
as a generalization of Litonski’s law [32] to 3D problems. Equation (4.67) has been used in Refs [51,52] to analyse asymptotic structure of propagating ASBs. All boundaries of the block are thermally insulated, its vertical edges traction free and the upper and the lower smooth horizontal surfaces are moved vertically in opposite directions to induce a nominal strain rate of 5000 s\(^{-1}\).

### 4.4.4 The h-Refinement

Figure 4.9 depicts the initial coarse mesh at time \(t = 0\) and the generated refined meshes at an average strain or non-dimensional time \(t = 0.025, 0.040\) and 0.047. In the solution of the problem, the mesh was also adaptively refined at \(t = 0.015, 0.030\) and 0.035; however, these are not shown for the sake of brevity. The times

![FE meshes at time t = 0.0 (A), 0.025 (B), 0.040 (C) and 0.047 (D). Source: From Ref. [42].](image-url)
for mesh refinement were manually selected and are arbitrary. A possibility is to refine the mesh when either \( I \) or the temperature at the specimen centre has risen by a certain amount. Meshes shown in Figure 4.9 vividly reveal that the refinement technique described in Section 4.2 generates non-uniform meshes with small elements in the severely deforming region and large elements elsewhere. No restriction was imposed on the number of new nodes that can be introduced during the refinement process. The distribution of the velocity field in the deforming region at \( t = 0.047 \), shown in Figure 4.10A, supports Tresca’s [11] and Massey’s [12] assertions that the tangential velocity is discontinuous across an ASB. In our work, the velocity field is forced to be continuous throughout the domain. The sharp jumps in the velocity components across the ASB lend credence to the discontinuity of the tangential velocity across the ASB. The plot of the effective stress, \( s_e \), defined as

\[
s_e = \sqrt{\frac{2}{3}} (1 - \nu \theta) (1 + b I)^m
\]

in Figure 4.10B reveals that \( s_e \) drops considerably within the ASB.

We now investigate the improvement, if any, in the approximate solution obtained by refining the mesh. Because the analytical solution of the problem is unknown, we compare the approximate solution with a higher-order approximate solution [53] obtained by smoothing out the computed solution. Let \( g \) be a solution variable to be smoothed. For the three-noded triangular element, we write

\[
g(\xi, \eta) = a \xi + b \eta + c \quad (4.69)
\]

Figure 4.10 At \( t = 0.047 \), distribution of (A) velocity field and (B) effective stress. Source: From Ref. [42].
where $\xi$ and $\eta$ are area coordinates of a point, and constants $a$, $b$ and $c$ are determined from values of $g$ at three quadrature points located in the triangular element. From Eq. (4.69), we evaluate $g$ at vertices of the triangle. Then the value $g^*_n$ of the smoothed solution at node $n$ is given by

$$g^*_n = \frac{1}{N_e} \sum_{n=1}^{N_e} g_n$$

(4.70)

where $N_e$ equals the number of elements sharing the node $n$. Knowing $g^*$ at each node, we interpolate its value at any other point by using the FE basis functions and define the percentage error $\eta$ in $D$ by

$$\eta = \left( \frac{||e||^2_0}{||e||^2_0 + ||\bar{D}||^2_0} \right)^{1/2} \times 100$$

(4.71)

where

$$e = \bar{D} - \bar{D}^*, \quad ||e||^2_0 = \sum_{e=1}^{N_{el}} \int_{\Omega_e} e^T e \, d\Omega$$

(4.72)

and $\bar{D}^*$ is computed from the smoothened velocity field. The plot of $\eta$ in Figure 4.11 for the three meshes shows that the error is lower for the approximate solution obtained by using the adaptively refined mesh than that for the other two meshes. The

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**Figure 4.11** Comparison of the error in the computed approximate solution and the higher-order approximate solution for three different meshes.  
*Source: From [42].*
error measure $\eta$ is not robust as indicated by slightly larger errors obtained with a fixed mesh of 841 nodes than those with 441 nodes. It could be due to the larger errors caused by smoothing out the approximate solution with 841 nodes because the band in this case is more intense than that for the mesh with 441 nodes.

4.4.5 The $r$-Refinement

Figure 4.12A shows the initial mesh having 400 uniform elements and Figure 4.12B, C and D show the generated refined meshes when the non-dimensional temperature $\theta$ at the plate centroid equalled 0.25, 0.35 and 0.45. The mesh was also refined when $\theta$ at the plate centroid reached 0.30 and 0.40, but these are not depicted. We chose to refine the mesh for equal increments of the temperature rise. However, other criteria

![Figure 4.12](image)

**Figure 4.12** (A) Initial mesh of uniform 400 square elements and FE meshes (B,C,D) generated using the $r$-refinement technique when the temperature rise at the specimen equalled 0.25, 0.35 and 0.45.

*Source:* From Ref. [43].
such as \( I \) or \( \theta \) attaining preassigned values would be equally as good. The meshes shown in Figure 4.12 vividly reveal that the \( r \)-refinement technique generated a non-uniform mesh with small elements in the severely deformed region and large elements elsewhere. The mesh smoothing criterion in Eq. (4.66) was applied at most three times to satisfy the requirement that the interior angles of every quadrilateral element be between 20° and 160°. We note that the mesh-generation scheme does not impose any restriction on the ratio of the area of the largest to that of the smallest element in the mesh. Even though this did not generate an unduly skewed mesh for the present problem, in other situations, such a restriction may be necessary. One could avoid this either by having more elements in the initial mesh or by adding elements at a few intermediate stages. The latter would necessitate the creation of a new mesh topology.

The percentage error in \( D \) is plotted in Figure 4.13. The error for the solution obtained by using the adaptively refined mesh is lower until an average strain of 0.037, and it then suddenly increases and becomes larger than that obtained for the solution computed with a fixed mesh. It is probably due to large errors caused by smoothing out the approximate solution in the late stages of the band development when deformations within the band are very intense.

### 4.5 ASBs and Cracks in Microporous Thermo-Viscoplastic Solids

#### 4.5.1 Problem Formulation

It is assumed that the material is microporous and exhibits thermal softening, strain and strain-rate hardening. In the referential description of motion, Eqs (4.1) and (4.2) are replaced by:

\[
\text{Balance of mass: } (\rho J (1 - f)) = 0
\]
Balance of linear momentum: \( \rho_0 (I - f_0) \dot{v} = \text{Div} T \) (4.74)

where \( f_0 \) (\( f \)) equals the volume fraction of voids in the reference (current) configuration. We assume that the strain-rate tensor has additive decomposition into elastic \( D^e \), plastic \( D^p \) and thermal parts \( \alpha \dot{\theta} \mathbf{I} \):

\[
D = D^e + D^p + \alpha \dot{\theta} \mathbf{I}
\] (4.75)

We postulate the constitutive relations

\[
\mathbf{\tau} \equiv \mathbf{\dot{\sigma}} + \mathbf{\sigma} \mathbf{W} - \mathbf{W} \mathbf{\sigma} = \frac{E(1 - f)}{1 + \nu} D^e + \frac{E(1 - f)}{(1 + \nu)(1 - 2\nu)} \text{tr}(D^e) \mathbf{I}
\] (4.76)

\[
\frac{3}{2} \frac{\text{tr}(s s^T)}{\sigma_m^2} + 2 f^* \beta_1 \cosh \left( \frac{\beta_2 \text{tr} \mathbf{\sigma}}{2\sigma_m} \right) - 1 - \beta_2 f^{*2} = 0
\] (4.77)

\[
D^p = \frac{(1 - f)\sigma_m^p \dot{\epsilon}_m}{\text{tr}(\sigma N^T)} \mathbf{N}, \quad s = \mathbf{\sigma} - \frac{1}{3} (\mathbf{tr} \mathbf{\sigma}) \mathbf{I}
\] (4.78)

\[
\mathbf{N} = \frac{3s}{\sigma_m^2} + \frac{f^* q_1 q_2}{\sigma_m} \left[ \sinh \left( \frac{q_2 \text{tr} \mathbf{\sigma}}{2\sigma_m} \right) \right] \mathbf{1}
\] (4.79)

\[
f^* = \begin{cases} f, & \text{if } f \leq f_c \\ f_c + \frac{f_u - f_c}{f_t - f_c} (f - f_c), & \text{otherwise} \end{cases}
\] (4.80)

\[
\sigma_m = \sigma_0 (1 + b \dot{\epsilon}_m^p)^m \left( 1 + \frac{\dot{\epsilon}_m^p}{\dot{\epsilon}_y} \right)^n (1 - \beta \dot{\theta})
\] (4.81)

\[
\dot{f} = (1 - f) \text{tr} D^p + \frac{f_i^2 \dot{\epsilon}_m^p}{s_2 \sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{\dot{\epsilon}_m^p - \dot{\epsilon}_N}{s_2} \right)^2 \right) H(-\sigma_{kk} - 0)
\] (4.82)

\[
\mathbf{q} = -k \left( 1 - \frac{3}{2} f \right) \text{grad} \theta
\] (4.83)

\[
\dot{\epsilon} = (1 - f_0) \epsilon \dot{\theta} + \frac{1}{\rho} \text{tr}(\mathbf{\sigma} (\mathbf{D} - D^p))
\] (4.84)
where

\[ T = J\sigma(F^{-1})^T, \quad Q = JF^{-1}q, \quad \sigma_m^2 = \frac{3}{2} S_{ij}S_{ij} \]  \hspace{1cm} (4.85)

\[ 2W = \text{grad} \: v - (\text{grad} \: v)^T, \quad (\dot{\varepsilon}_m^p)^2 = \frac{2}{3} D_{ij}^p D_{ij}^p \]  \hspace{1cm} (4.86)

Here, \( E, \nu \) and \( \alpha \), respectively, are Young’s modulus, Poisson’s ratio and the coefficient of thermal expansion for the matrix material; \( I \) is the unit tensor. We refer the reader to Refs [54–57] for detailed descriptions of these equations. Nearly all of the thermo-physical material parameters depend on the temperature; however, such dependencies have been neglected for the sake of simplicity. While homogenizing material properties of a particulate composite, the rule of mixtures is used.

### 4.5.2 Plane-Strain Shearing of an Inhomogeneous Plate

We discuss results for plane-strain shearing of a plate of thickness \( 2H \) with all sides thermally insulated, vertical edges traction free and equal and opposite tangential velocities applied to the top and the bottom surfaces as depicted in Figure 4.14. The prescribed speeds increase from zero to the steady value \( \nu_0 \) in time \( t_r \), giving an eventual nominal shear strain rate of \( \nu_0 / H \).

An ASB and a crack are assumed to initiate at a point when the effective plastic strain there equals 1.5; the difference between simulating the two is that for an ASB, no separation between adjoining surfaces occurs, but for a crack, the two surfaces slide relative to each other. The latter is simulated by using the node-splitting technique. The FE mesh used to analyse the problem had elements of size 4.6 \( \mu \text{m} \times 4.6 \mu \text{m} \) in the central and end portions but are larger in other regions. For inhomogeneous bodies, the composition varied from 100\% W (tungsten) at the centroid to 0\% W at the edges. The yield stress of the material of elements in the 0.2 mm long, 4.6 \( \mu \text{m} \) thick layers located symmetrically about the centroidal axis was reduced by 30\% to nucleate an ASB there. The ASB and the crack were found to propagate horizontally to the left and to the right with equal and opposite velocities; thus, their propagation to the right is described here. Results taken from Refs [58,59] presented next are for \( t_r = 1 \mu \text{s} \) and \( \nu_0 = 0.625 \text{ m/s } \) giving \( \dot{\gamma}_{\text{avg}} = 5000 \text{s}^{-1} \).

Figures 4.15 and 4.16 depicts variations of the crack and the ASB lengths as they advance in the two homogeneous materials, W and NiFe (nickel–iron), and the

![Figure 4.14 Schematics and FE mesh for plane-strain shearing of a plate.](image-url)
inhomogeneous material W2NiFe. Because of the compressive normal stress acting on the crack surfaces, they always contact each other. In W, time histories of the crack length and the ASB length are virtually identical. However, in NiFe, the crack length is larger than the ASB length, and the crack propagation speed is higher than the ASB speed.

Except when the crack arrives at the edges, its speed increases from about 0.1 km/s at initiation to about 1.8 km/s when computations were terminated. The crack accelerates slowly in the beginning but quite rapidly towards the end. An ASB/crack initiates in NiFe at 165 \( \mu \text{s} \) but in W at 35 \( \mu \text{s} \). Interestingly, plots of the crack propagation speed versus the crack length for W and NiFe are nearly coincident; cf., Figs 4.16A,B. We note that the Rayleigh wave speeds in W and NiFe equal 2.61 and 3.02 km/s, respectively. For the W2NiFe plate, the effective plastic strain at a point reaches 1.5 at about 46 \( \mu \text{s} \). The plot of the crack speed versus the crack length (cf., Fig. 4.16C) is quite different from that for either W or NiFe, and

Figure 4.15 Time histories of evolution of a crack and an ASB in (A) W, (B) NiFe and (C) W2NiFe plates deformed in plane-strain shear. 
Source: From Ref. [58].

Figure 4.16 Variation with crack/ASB length of the crack/ASB speed in (A) W, (B) NiFe and (C) W2NiFe plates deformed in plane-strain shear. 
Source: From Ref. [58].
the maximum crack speed achieved equals about 0.14 km/s, which is about one-tenth of that in W and NiFe. Also, except immediately after the crack and the ASB initiate, the ASB speed is less than that of the crack.

Time histories of the tangential force required to deform the specimen are exhibited in Figure 4.17. Recalling that the top and the bottom surfaces are constrained from moving in the vertical direction, the working of external forces is due to tangential traction and is proportional to it because the tangential velocity, except for the first $1 \mu s$, is constant. The tangential force decreases with the opening of a crack and continues to decrease as the crack elongates. With the crack extension, a smaller surface area supports the external load. Also, with continued plastic deformations, the material softens, and its capacity to support external load diminishes. The rate of decrease of the applied tangential force is higher for W than that for NiFe because the strain and strain-rate hardening effects are higher in NiFe than those in W. With the extension of the crack, the driving force drops more rapidly for the W2NiFe plate than that for the W and NiFe plates.

The hydrostatic pressure in the material within the ASB, ahead of it and directly in front of the crack tip, was found to be compressive. Thus, no new voids nucleated in the material even though it had undergone enormous plastic deformations.

The homogenization of material properties for the W2NiFe plate suppressed sharp gradients in deformations likely to occur near interfaces between the matrix and the particulates. Furthermore, extensive plastic deformations during the ductile failure may debond particulates from the matrix, as discussed in the following section.

### 4.5.3 Plane-Strain Compression of a Particulate Composite Plate

We present results for plane-strain compression of a rectangular plate made of W circular cylindrical fibres immersed in NiFe matrix and explore effects of the interfacial bonding on the ASB initiation; a schematic sketch of the problem studied is shown in Fig. 4.18. The prescribed velocity increases linearly from zero.
to its steady value in $1 \mu s$ so that the maximum nominal axial strain rate equals $10^4$ s$^{-1}$. An ASB is assumed to initiate at a point when the energy dissipation rate there suddenly increases by nearly an order of magnitude, the material point is deforming plastically and deformations in its neighbourhood are inhomogeneous. For high strain-rate deformations of particulate composites, it was found [57] that the ASB initiation criterion stated in Section 4.3.2 is satisfied at a material point at time $t_1$ but not at time $t_2 > t_1$ due to load exchange between particulates and the surrounding matrix.

We use the cohesive zone model [60–63] to delineate interfacial failures between particulates and the matrix. After every time step, each interface between two adjoining FEs is checked for failure (i.e., separation and/or sliding) by examining the state variables of the two elements. The traction–separation relation is taken to be triangular with the area of the triangle representing fracture energy per unit surface area. The traction first linearly increases with an increase in the separation between the two faces of an interface to a pre-assigned maximum value and then affinely decreases to zero when the separation reaches a critical value. The mode mixity is considered by following the approach suggested by Ortiz and Pandolfi [63].

When interfaces are not allowed to separate, ASBs formed with plastic strains of 100% and a nearly discontinuous velocity field along surfaces inclined at about 45° to the loading axis. Figure 4.19 shows fringe plots of the effective plastic strain and the vertical component of the velocity. It is clear that one dominant ASB formed and divided the region into two parts: the lower essentially stationary pyramid and the upper one moving downward with the velocity prescribed on the top

![Figure 4.18 Schematic sketch of plane-strain compression of a particulate composite plate.](image-url)
surface. There is a sharp gradient in the velocity field between these two regions where strain rates, the effective plastic strain and the temperature are very high. In Figure 4.19B, points 1 and 3 are in the ASB, and points 2 and 4 are outside the ASB; points 1 and 2 are in the matrix, and points 3 and 4 are in the particulates. Time histories of the effective plastic strain rate exhibited in Figure 4.20 reveal

Figure 4.19 For plane-strain compression of a particulate composite plate: (A) fringe plots of the effective plastic strain at $t = 22.5 \, \mu s$ and (B) plots of the vertical component of velocity. 
Source: From Ref. [59].

Figure 4.20 Time histories of the effective plastic strain rate at four points marked in Figure 4.19B. 
Source: From Ref. [59].

...
that the effective plastic strain rates at these four points exceed $10^4\,\text{s}^{-1}$, except at late times when the ASB had formed. The strain rates at points 1 and 3 increase by a factor of 10 as the ASB develops.

Fringe plots of the $x_2$-velocity depicted in Figure 4.21 are for the simulations with failure allowed at interfaces. Dark circles denote regions where the matrix has separated from the particulates, and white regions denote voids and cracks. The deformed shapes and locations of ASBs strongly depend on the mode-mixity parameter, $\beta$. A smaller value of $\beta$ makes the interface more susceptible to shear fracture; for $\beta = 0.6$, many particulate/matrix interfaces are fractured prior to the ASB development that result in totally different stress states from when there is no debonding. The ASB initiation time for $\beta = 0.6$ is nearly one-half of that for $\beta = 1.1$. Thus, the mode mixity and the cohesive failure criterion play dominant roles in not only the formation of ASBs but also in their spatial locations.

4.6 Concluding Remarks

As should be clear from results discussed earlier, 1D and 2D problems can be analysed with fine and properly graded meshes to obtain essentially mesh-independent results that are usually controlled by the thermal length scale. Very few 3D problems involving the localization of deformation into narrow bands have been scrutinized. Batra and Zhang [64] used the large-scale explicit FE code DYNA3D to analyse torsional deformations of a thin-walled tube and computed the speed of ASBs initiating from weak elements near the tube’s centre and propagating outwards circumferentially. The speed strongly depended on the nominal strain rate; at an average strain rate of $5000\,\text{s}^{-1}$, it varied from 180 m/s at the instant of initiation to about 900 m/s by the time it reached the opposite end. Batra and Rattazzi [65] also used DYNA3D to study torsional deformations of a thick-walled tube and found that an ASB initiating from the root of a V-notch at the tube’s centre propagated in the radial direction.
at about 100 m/s. However, when the thermal softening of the material was modelled by a different function that enhanced the effect, the speed of the shear band in the radial direction increased to about 1000 m/s. During the simulation of the Taylor impact test for a tungsten heavy alloy (WHA) rod, Batra and Stevens [66] found that the speed of an ASB (a contour of effective plastic strain of 1.0) initially equalled 800 m/s in the axial direction and 550 m/s in the radial direction and quickly dropped to 150 m/s in each direction.

Zhu and Batra [67] studied the possibility of phase transformation within an ASB. In the undeformed state, the specimen was assumed to be fully annealed and isotropic, and its microstructure was a mixture of coarse ferrite and cementite. A material point was assumed to transform into austenite after its temperature exceeded the transformation temperature with the rate of transformation governed by a simple kinetic equation. Proper account was taken of the latent heat required for the transformation, the associated volume change and the variation in the thermo-physical properties. It was found that the austenite is quenched rapidly enough by the surrounding material for it to be converted into martensite rather than a mixture of pearlite and martensite.

Wang and Batra [68] accounted for the texture development by using a theory with two internal variables, a scalar to account for the isotropic hardening of the material and a symmetric traceless second-order tensor to account for the kinematic hardening. They found that the consideration of kinematic hardening does not alter the qualitative nature of results.

It is hoped that this article has elucidated upon some aspects of ASBs and has raised enough questions in the reader’s mind to warrant further enquiry into the subject. It will be interesting to find out if any new physics is revealed by studying 3D problems with adaptively refined meshes.

References


